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THE STEREOSCOPE

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INTRODUCTION

THE stereoscope shares with the kaleidoscope the distinction of having been the two scientific instruments in the last century which obtained the greatest popularity. It is said that over one million kaleidoscopes were sold within a year of its invention, and in the 'sixties the stereoscope attained almost equal popularity. Up to the invention of wireless, nothing scientific has ever, I believe, so gripped the imagination of the people. The cause is not far to seek—both of these instruments produced by very simple means very beautiful and fascinating results.

The stereoscope is based upon the observation that the image that is projected upon the retina of the right eye of a solid object, or of objects at different distances from the observer, is slightly different from the image that is projected upon the retina of the left eye. Exactly how these slightly different pictures are fused and interpreted by the brain has given rise to a great deal of discussion, and even now there is no unanimity on the matter.

Brewster* stated that an attempt to explain the appearance of stereoscopic relief was given in 1613 by Aguilonius who said:

Therefore, since the images in each eye are dissimilar (*minime sibi congruunt*) the representation of the object must appear, confused and disturbed (*confusa ac perturbata*) to the primary sense.

This view of the subject is certainly consistent with reason, but, what is truly wonderful is, that it is not correct, for bodies are seen clearly and distinctly with both eyes when the optic axes are converged upon them. The reason of this, I think, is that the bodies do not appear to be single, because the apparent images, which are formed from each of them in separate eyes, exactly coalesce (*sibi mutuo exacte congruunt*), but because the common sense

* D. Brewster, *The Stereoscope*, p. 12.

imparts its aid equally to each eye, exerting its own power equally in the same manner as the eyes are converged by means of their optical axes. Whatever body, therefore, each eye sees with the eyes conjoined, the common sense makes a single notion, not composed of the two which belong to each eye, but belonging and accommodated to the imaginative faculty to which it (the common sense) assigns it.

Aguilonius, however, is not here referring to stereoscopic relief at all, but to the inequality of the images of any plane figure which is looked at obliquely by the two eyes simultaneously, as he explains in the two paragraphs which precede the ones which Brewster translated; in these paragraphs he shows how the rays proceeding from a circle, triangle, or other plane figure to the eye form a cone or pyramid having its vertex at the centre of vision of the eye. In fact, if such a figure is looked at obliquely at a distance of about 10 inches, there is a difference of about 2 per cent. in the sizes of the images. That this difference is not noticeable may be explained perhaps by the "common sense" of Aguilonius.

Earlier writers had noticed the dissimilarity of the images of solid bodies; Euclid (c. 300 B.C.), Galen (c. 150 A.D.), Ptolemy (c. 150 A.D.), Leonardo da Vinci (*Trattato della Pittura*, c. 1584), and Porta (*De Refractione*, 1593, pp. 133, 143) have all mentioned it. Da Vinci pointed out that parts of distant objects hidden from one eye by a small near object become visible upon opening the other eye.

Many writers also knew that a concave mirror produced a real image, and described how to form an image apparently actually in the air. For instance, Porta describes in his *Magia Naturalis*, 1558, how to form such an image, which appeared to the spectator to be formed in the air, in front of him. This experiment can easily be repeated, and it is astonishing to anyone performing it for the first time how real this image can be made to appear. I do not suggest, of course, that the mirror used by Porta was equal to one that would be used to-day, and therefore the image he formed would no doubt have appeared somewhat confused. Kepler (*Ad Vitellionem*, 1604), Magini (*Breve Istruttione . . . dello specchio concavo sferico . . .*, 1628), Kircher (*Ars Magna Lucis et Umbrae*, 1646), and many others described similar experiments.

Professor von Rohr has drawn my attention to two experiments described by Dr R. Smith in *A Compleat System of Opticks*, 1738 (vol. 2, p. 388, and Remarks, pp. 526, 527), which can easily be set up*. In the one he draws on a piece of paper two lines making a small angle, one with another, which he looks at obliquely. The one line is drawn along the intersection of the paper and a vertical plane through the observer's right eye, and the other line in the corresponding intersection of the paper and a vertical plane through his left eye. Then he noticed that the lines appeared to fuse into one which seemed to be standing up vertically, as though there were a pin stuck in the paper. In the other experiment:

He opened a pair of compasses, and while he held the joint in his hand, and the points outwards and equidistant from his eyes, and somewhat higher than the joint, he looked at a more distant point; the compasses appeared double. He then compressed the legs until the two inner points coincided; having done this the two inner legs also entirely coincided,

* Models illustrating these experiments were passed round the room.

and bisected the angle formed by the outward ones, appearing longer and thicker than they did, and reaching from the hand to the remotest object in view. The explanation offered by Dr Smith accounts only for the coincidence of the points of the compasses, not for that of the entire leg. The effect in question is best seen by employing a pair of straight wires about a foot in length. A similar observation, made with two flat rulers, and afterwards with silk threads, induced Dr Wells to propose a new theory of visible direction in order to explain it, so inexplicable did it seem to him by any of the received theories. (Wheatstone, *Scientific Papers*, 1879, p. 237; see also Smith, *System of Opticks*, vol. 2, p. 388, and Remarks, p. 526.)

Harris, in his *Treatise on Optics* (1775, p. 171), says:

We have other helps for distinguishing prominences of small parts besides those by which we distinguish distances in general, as their degrees of light and shade, and the prospect we have round them. And by the parallax, on account of the distance betwixt our eyes, we can distinguish besides the front part of the two sides of a near object not thicker than the said distance, and this gives a visible relief to such objects, which helps greatly to raise or detach them from the plane in which they lie. Thus the nose on a face is the more remarkably raised by our seeing both sides of it at once.

WHEATSTONE'S INVENTION OF THE STEREOSCOPE

Thus the fact that the images seen by the two eyes are slightly different was known and must have been observed by a great many people, but it can hardly be said that any explanation of the stereoscopic appearance had been given. It had certainly never occurred to anyone that an appearance of a solid object in relief could be obtained artificially by presenting two slightly different drawings of the object to the right and left eye respectively, and it was Wheatstone who first conceived this idea. In many cases an invention is the result of a number of steps taken by different individuals and the way is thus prepared for the actual invention. In this case the whole credit of the invention, however, belongs to Sir Charles Wheatstone.

Wheatstone, in June 1838, in his communication to the Royal Society entitled "Contributions to the Physiology of Vision," described how to produce the illusion of stereoscopic relief by the artificial combination of two dissimilar plane representations of an object. That the possibility of this achievement had escaped earlier recognition he ascribed

to this circumstance—that the results being contrary to a principle which was very generally maintained by optical writers, viz. that objects can be seen single only when their images fall on corresponding points of the two retinæ, an hypothesis which will be hereafter discussed, if the consideration ever arose in their minds, it was hastily discarded under the conviction that if the pictures presented to the two eyes are under certain circumstances dissimilar, their differences must be so small that they need not be taken into account.

Wheatstone describes a number of experiments on stereoscopic vision of which I will mention one or two later. He discusses one obvious explanation and shows its inefficiency. He says:

It may be supposed that we see but one point of an object distinctly at the same instant, the one namely to which the optic axes are directed, while all other points are

seen so indistinctly, that the mind does not recognise them to be either single or double, and that the figure is appreciated by successively directing the point of convergence of the optic axes successively to a sufficient number of its points to enable us to judge accurately of its form.

That there is a degree of indistinctness in those parts of the field of view to which the eyes are not immediately directed, and which increases with the distance from that point, cannot be doubted, and it is also true that the objects thus obscurely seen are frequently doubled. It may be said, this indistinctness and duplicity is not attended to, because the eyes shifting continually from point to point, every part of the object is successively rendered distinct; and the perception of the object is not the consequence of a single glance, during which only a small part of it is seen distinctly, but is formed from a comparison of all the pictures successively seen while the eyes are changing from one point of the object to another.

That this is not the explanation he shows by another of his experiments, and as this cannot be described better than in his own words, I will quote them:

Another and a beautiful proof that the appearance of relief in binocular vision is an effect independent of the motions of the eyes may be obtained by impressing on the retinæ ocular spectra of the component figures. For this purpose the drawings should be formed of broad coloured lines on a ground of the complementary colour, for instance red lines on a green ground, and be viewed either in the stereoscope or in the apparatus, fig. 6, as the ordinary figures are, taking care, however, to fix the eyes only to a single point of the compound figure; the drawings must be strongly illuminated, and after a sufficient time has elapsed to impress the spectra on the retinæ, the eyes must be carefully covered to exclude all external light. A spectrum of the object in relief will then appear before the closed eyes.

As in this case the pictures cannot shift their places on the retinæ in whatever manner the eyes be moved about, the optic axes can, during the experiment, only correspond with a single point of each.

Sufficient, however, has been shown to prove that the laws of binocular visible position hitherto laid down are too restricted to be true.

Actually he gives no explanation of his own.

Fig. 1 is a photograph of the original apparatus* which Wheatstone described to the Royal Society in 1838. This apparatus for the first time enables the observer to see, apparently superposed upon one another, two separate drawings of a solid object, which were presented to the right and left eye respectively; in effect he seemed to be looking at a single image of an object which, as we know, then appeared to be in stereoscopic relief. The superposition was obtained very simply by placing a plane mirror in front of each eye; the mirrors made angles of 45° with the line of sight, and were perpendicular to one another. The drawings faced one another, and were so arranged that the images formed by the mirrors were superposed. By means of a right and left-handed screw, which can be seen in the photograph, he could vary the distance of the drawings, always keeping them equidistant, so that their image planes should always coincide.

In the same paper Wheatstone described other methods of producing the effect. For instance, he arranged to look directly at the one drawing and to superpose

* By the courtesy of the Delegates of King's College, at the kind request of Professor Appleton, this apparatus was lent for exhibition.

another drawing upon it by reflection at a single mirror. Also he showed that a single drawing—say of two circles struck from slightly different centres—looked at directly with one eye and by reflection in a plane mirror with the other eye, produced an appearance of stereoscopic relief.

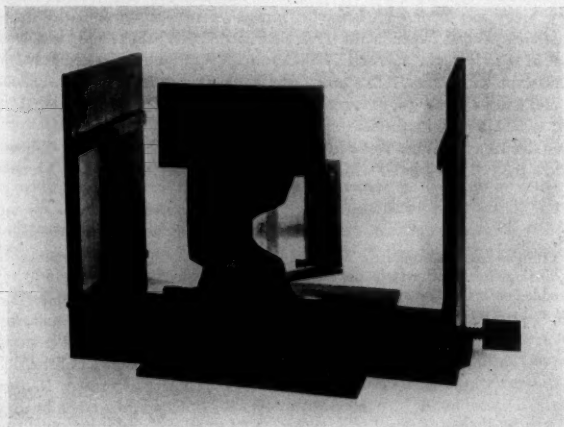


Fig. 1

Although Wheatstone's paper above referred to was actually read in 1838 he had investigated the stereoscopic image earlier, for, in his second paper communicated to the Royal Society in 1852, he says:

In 1833, five years before the publication of the memoir just mentioned, these yet unpublished investigations were announced, in the third edition of Herbert Mayo's *Outlines of Human Physiology*, in the following words: "Mr Wheatstone has shown, in a paper he is about to publish, that if by artificial means the usual relations which subsist between the degree of inclination of the optic axes and the visual angle which the object subtends on the retina be disturbed, some extraordinary illusions may be produced. Thus, the magnitude of the image remaining constant on the retina its apparent size may be made to vary with every alteration of the angular inclination of the optic axes."

This is of importance in assigning the invention to Wheatstone and proving that he had preceded Elliott, a Glasgow schoolmaster to whom Brewster tried to attribute the priority, in combining a pair of different stereoscopic pictures. In his instrument* Elliott limited the view presented to each eye by arranging that the observer should see the two pictures through a box, which had a square aperture at the bottom of such a size that the right eye only saw the left-hand picture, and the left

* A model of this instrument was passed round the room.

eye only saw the right-hand picture through this aperture. The eyes, of course, in this case had to be converged beyond the normal amount.

Brewster, who appears to have been jealous of the recognition given to Wheatstone for his discovery and anxious to belittle it, tried to show firstly that Wheatstone was anticipated by Elliott and secondly that Wheatstone's explanation (which he did not give) was incorrect. Brewster himself, however, says that Elliott read Wheatstone's second paper in the *Phil. Mag.* in March, 1852, and then wrote to say that in 1839 he had constructed an instrument by which he looked at a drawing of a landscape with moon, hill, a cross, and a branch of a tree, which were seen in relief, and that he showed it to certain friends in Liverpool; he affirmed that he had *thought of* making the instrument as early as 1834!

Brewster attacked Wheatstone in *The Times* in 1856, and in his reply Wheatstone published the following letter which he obtained from R. Murray of "Murray and Heath":

43, PICCADILLY, W.

October 27th, 1856.

SIR,

From an examination of the accounts furnished to you by Mr Newman of Regent Street, during the time I was in his establishment and which were prepared by myself, I am enabled to fix the date of my first knowledge of your stereoscopes both with reflecting mirrors and refracting prisms to the latter part of 1832.

I am,

Yours faithfully,

R. MURRAY.

This letter, as Professor von Rohr pointed out to me, was published again in the *Brit. Journ. Phot.* of 4th Feb. 1910. From this letter it is quite clear that Wheatstone had used both prisms and mirrors in his stereoscopes as early as 1832. Brewster also tried to make out that he himself was the first one to use photography for stereoscopic work, but this is not the case, for in his second paper to the Royal Society in 1852 Wheatstone says:

at the beginning of 1839...the photographic art became known and soon after at my request, Mr Talbot, the inventor, and Mr Collen (one of the first cultivators of the art) obligingly prepared for me stereoscopic Talbotypes of full-size statues, buildings, and even portraits of living persons.

Wheatstone sent specimens to M. Quetelet, who mentioned the fact in the *Bulletin of the Brussels Academy* of October, 1841. Having of course read all this, Brewster remarks:

It is singular that neither Mr Elliott nor Mr Wheatstone should have availed themselves of the well-known photographic process of Mr Wedgewood and Sir Humphrey Davy....

Wheatstone's Talbotypes, therefore, anticipated Brewster's Daguerreotypes by about ten years, which fact Sir David decided to ignore though he must have been fully aware of it.

Wheatstone had seen that this new process would afford far better results than

were possible by any drawing owing to the accuracy with which the camera would produce the required image. He afterwards induced Claudet to take some photographs for him for this purpose. Claudet (1797-1867) was one of the first of those who realised the value of Daguerre's achievement. He had originally come to London to introduce the productions of the Choisy glass works. On the introduction of photography he was immediately interested and turned his energies to it with enthusiasm. Among other things he introduced the use of the bromide and chloride of iodine for accelerating the production of the Daguerreotype image, by which it was made a hundred times more rapid, and he became a most skilful exponent of the Daguerreotype process. The decline of this beautiful process was due to the invention of the collodion film on glass by Scott Archer. Claudet, having assisted Wheatstone in the application of photography to the stereoscope, continued to take pictures for it commercially. He explained its scientific principles and published many papers dealing with this and with other matters in connection with photography, for instance, dark room illumination, the difference of the actinic and visual foci of a lens, the effect of the atmosphere on exposure (1849), the angle to be given to binocular pictures, the effect of the position of the masks in enhancing the stereoscopic effect (1857).

Brewster, in a further attempt to deprive Wheatstone of the invention, in 1862 put forward a pair of oil portraits about $12 \times 8\frac{1}{2}$ inches, which the Rev. Crum Brown had found in the Museum of Wiess at Lille, signed by Jacopo Climenti (1554-1640), as having been intended to be combined stereoscopically. The portrait is that of a man sitting on a stool, holding a piece of string in one hand. Treated as a pair of stereoscopic pictures Professor Emerson of Toronto found by measurement that the man's back would be nearer than his near shoulder and that his far foot would be nearer than his nearer one. In fact, owing to slight variations in the drawings, it was a matter of accident as to which part of his anatomy would be stereoscopic and which part pseudoscopic. The one picture is also slightly larger than the other. There can be little doubt that the one is merely a careful copy of the other, and in any such case a stereoscopic combination will cause parts of the picture to seem in relief due to accidental displacement.

As Brewster's is probably the best known book on the stereoscope, and as it is so wilfully misleading with regard to Wheatstone's share in its invention, and the case for Wheatstone is to a large extent buried in the old journals and, therefore, not known to the present generation, I think it is only fair to his memory to have introduced these few remarks. I would only add that Professor von Rohr has pointed out (*Die Binocularen Instrumente*, p. 54) that Wheatstone in 1842 had intended to have given further lectures as a continuation of his first contribution.

OTHER FORMS OF STEREOSCOPE DEvised BY WHEATSTONE

Wheatstone realised the disadvantages of his original form of stereoscope; two portable forms of Wheatstone's mirror apparatus* are shown in Fig. 2. In the one on the left of the photograph the arms carrying the pictures are hinged to the

* These were exhibited.

central parts and after folding these together, the whole, with another hinge, closes inside the box. This form was apparently put on the market, as in addition to the one at King's College there is one also in the Science Museum. The second one shown in its folded up condition also hinges together, but the end screens holding the pictures fit on to the side arms with dowels.

Fig. 3 shows several interesting forms of Wheatstone's apparatus* from the Wheatstone Collection at King's College. On the right is a hand stereoscope in which the pictures are placed one above the other in grooves at the back of the

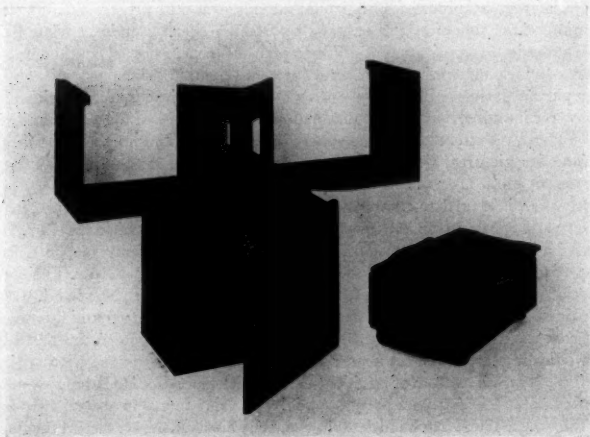


Fig. 2

wooden screen seen in the illustration. Two mirrors are, as shown in the illustration, mounted at the end of an arm, and are so inclined that the left eye sees the image of the upper picture superposed on the image of the lower picture seen by the right eye. In the foreground in the middle of the picture is another hand stereoscope, with curious prism eyepieces. In the background are two panoramic views—one above the other—which can be superposed by the aid of a pair of prism spectacles (seen in the bottom left-hand corner). Between these on the left is a prism binocular somewhat similar to Brewster's binocular. I do not know, however, what is the date of this. In the bottom right-hand corner are skeleton cubes and a pyramid made of square brass wire, to which I will refer when I am discussing his pseudoscope.

* These were exhibited.

If a pair of stereoscopic pictures is projected on to a screen*—one in red, the other in green light—and the confused superposed images are observed through coloured spectacles (one red and one green filter), the images are seen clearly by the right and left eyes respectively, and it is possible to combine the two pictures into one and see the object in stereoscopic relief. Then, if the observer fixes his attention on any one point of the picture, a neighbouring point which is closer to the observer is by some subconscious process seen as a single point, although actually the image thrown on one retina does not correspond with that thrown on

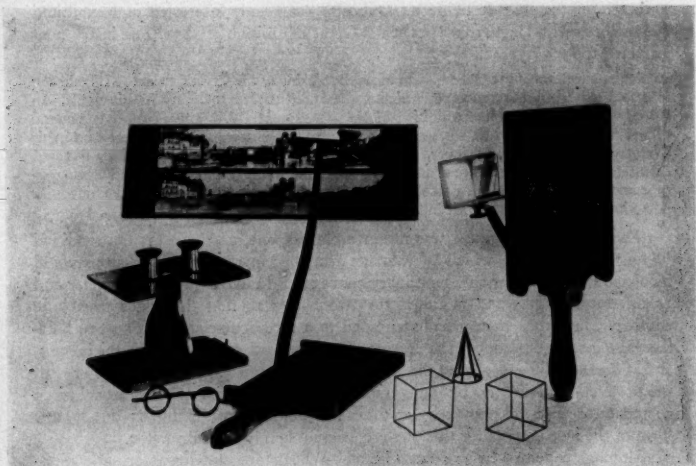


Fig. 3

the other retina, that is although the images of the two points are displaced relatively to one another in the one eye as compared with the images of the same points in the other. If the difference between the distances of the two objects which are nearly in a line with one another is not too great, I find that when I fix one of them, so long as I know the relative position of the other one, I see that other one as a single point, and if I then fix the second one the first one also remains single. I think that this result supports an explanation of stereoscopic vision similar to that given by Aguilonius for the fusion of images of unequal sizes.

Leaving this somewhat controversial and psychological aspect of stereoscopic vision, which is by no means settled as is evident from the papers of Dr French,

* A demonstration of this experiment was given, the audience being provided with coloured spectacles.

Commander Langlands, and Mr Trump in the recent *Transactions* of this Society, let me return to the early history of the subject.

BREWSTER'S LENTICULAR STEREOSCOPE

Brewster in 1849 constructed the lenticular stereoscope. He had some made by Loudon of Dundee, which he described to the Royal Society of Arts. He was unable to get the instrument taken up by opticians either in London or Birmingham, so he took one to Paris in 1850 and showed it to M. Soleil and to his son-in-law, Jules Duboscq, of the Rue de l'Odéon. Duboscq was keenly interested and exhibited stereoscopes with Daguerreotypes at the Great Exhibition of 1851. The instrument attracted the

particular attention of the Queen, and before the closing of the Crystal Palace, M. Duboscq executed a beautiful stereoscope, which I presented to Her Majesty in his name. In consequence of this public Exhibition of the Instrument, M. Duboscq received several orders from England, and a large number of stereoscopes were thus introduced into this Country. The demand, however, became so great that opticians of all kinds devoted themselves to the manufacture of the instrument, and photographers, both in Daguerreotype and Talbotype, found it a most lucrative branch of their profession, to take binocular portraits of views to be thrown into relief by the stereoscope (Brewster, p. 31).

The reflecting apparatus was rather a clumsy one, and even in the modified folding form of Fig. 2, which Wheatstone described in 1852, it still was bulky and inconvenient. Brewster cut an ordinary convex lens in halves (10 cm. diameter and 10 cm. focal length) and mounted the two halves so that the edges faced one another. The lenses were usually trimmed to a rectangular shape. The two half-lenses so mounted produced the necessary prismatic effect to superpose the images, and at the same time magnified the picture. This simplified apparatus, together with the production of pictures by photography, made the stereoscopes a practical piece of apparatus, and the popularity of the stereoscope was undoubtedly due to this form of the instrument; for this due credit must be given to Brewster.

Although invented by two Englishmen—Wheatstone and Brewster—the apparatus was thus first developed in France. Duboscq took out a patent dated 16th Feb. 1852, and made the instrument in great numbers; he also supplied photographs for use with them—both Daguerreotype and transparencies. For the latter he fitted the stereoscope with a ground glass plate at the far end. Other French firms quickly joined in to participate in the trade that resulted.

Duboscq brought an action against M. Gaudin for infringement of his patent which was heard on appeal in the Imperial Court of Paris in 1858. It was stated by Duboscq's counsel that Sir David Brewster had brought his stereoscope to Duboscq, who, after examination, had improved upon it by substituting for the wooden bottom a piece of ground glass, and had also made other alterations. M. Gaudin pleaded that the invention was not new; also that it had been published in the *Revue Scientifique* previous to its having been patented, and had, therefore, been given to the public. The Court overruled these objections, declared the seizure of the articles made at the instance of Duboscq to be good, and directed that all

those described in the *procès-verbal* should be given up to him, gave Duboscq damages, and condemned M. Daudin to pay the costs.

Duboscq painted the back of his transparencies with a matt varnish and he brought an action against M. Ferrier for an infringement of this patent; this, however, he lost. To show the amount of the trade at that time, it may be mentioned that the stock of M. Ferrier which was seized pending this action was valued at no less than 50,000 francs.

LATER FORMS OF STEREOSCOPES

A great many patents were taken out, both in England and abroad, from 1850 onwards*. During the craze a demand sprang up for private stereoscopic portraits, and a number of pocket stereoscopes were produced†.

Among the many varieties introduced I may mention a reflecting stereoscope by Horn and Thornthwaite; a bijou stereoscope and slides, which is the first midget stereoscopic outfit made; a simple hand reflecting stereoscope; Brown's simple reflecting stereoscope; a stereoscope by Duboscq which opens and closes like an opera hat (1843). All of these are from the Court collection.

A very interesting and beautiful, though somewhat expensive, one was Swan's "Crystal Cube," 1862. It is in the form of a rectangular block of glass built up from two right-angled prisms which are ground to an angle of about 39° or 40°, and the two pictures are mounted on two of the outer faces of the resulting block. If the angle of the prisms is rightly adjusted, the critical angle will be so situated that one of the observer's eyes comes within and the other comes beyond the critical angle. Then, owing to the internal reflection by the air film at the hypotenuse face, the one of these pictures is seen by the right eye by reflection, the other one by the left eye by transmission, and the resultant image appears to be in relief inside the block of glass. It is obvious that one of the pictures must be reversed.

Swan also invented what he called a "Clairvoyant" stereoscope; the special feature was the use of corrugated glass between the two lenses. There is really not much point in this, as the result can be obtained equally well in a simpler manner.

Another rather interesting stereoscope is that of Beck, invented in 1859, which was intended to be used to examine pictures mounted in any way, for instance, in a book; there is no box and there is a reflector to aid the illumination of the picture looked at. The screen to separate the picture is also a translucent one for the same purpose.

Beck, also in 1859, pointed out the importance of the position of the masks, in making the view appear at a distance as though seen through a window, but, as we have seen, this had already been recommended by Claudet in 1857.

There are two important adjustments necessary to enable the observer to see stereoscopic pictures effectively: (1) he should be able to focus the image to suit

* Examples of all the stereoscopes referred to (except that of Maxwell) from the Court collection, the Science Museum, and my own collection were exhibited.

† An early and rare form of pocket stereoscope by Claudet, representing the highest class, and another smaller but very fine example by Kilbourne were exhibited.

his sight, and (2) he should be able to alter the separation between the lenses, both to compensate for the difference in the interpupillary distances in different individuals and for the variations in the distance apart at which the pictures have been mounted. Usually these adjustments were provided by the lenses being cut to circular form and independently mounted.

In 1867 James Clerk Maxwell invented a "New Stereoscope" which was rather interesting. At one end of a board about 2 feet long he placed a transparency upside down. Approximately in the middle of the board he placed two lenses, each having a focal length of about 6 inches, which therefore produced a real, erect image of the transparency at the other end of the board. Here he placed a large lens having a focal length of about 8 inches. To an observer 2 or 3 feet farther away, the object was apparently produced as a solid object situated in the neighbourhood of the large lens.

Still more complicated instruments were made in which extra lenses could be introduced, such as the "Magic Stereoscope" of Negretti and Zambra, in which a pair of convex lenses mounted on an axis beyond the viewing lenses can be thrown into the optical path, or a pair of similarly mounted concave lenses could be inserted. It has also an extra pair of large convex lenses some 3 inches from the viewing lenses. The slide can be put either 3 or 6 inches from the viewing lenses.

In another stereoscope called a "Cosmorama Stereoscope," the top door which admits the light for viewing opaque slides is connected to the end door which admits light to transparencies. Thus one closes as the other opens, and a change from "daylight" to "night" is produced.

Achromatised lens systems were supplied for some of the expensive instruments.

Quite a number of patents referred to colouring the slides, by introducing coloured glass in the incident light or elsewhere in the optical system*.

THE PSEUDOSCOPE

Wheatstone, in his first paper, 1838, § 5, pointed out that by interchanging the pictures presented to the right and left eyes, an inverse stereoscopic result was produced which he termed "pseudoscopic." The inversion is very easily obtained with simple line drawings, and can also be obtained fairly easily in the case of medallions, embossed trays, and other objects in which we are accustomed to the inversion of the relief. It is obvious, however, that the psychological factor comes in powerfully, for I have found that there are quite a number of people who are unable to obtain the pseudoscopic illusion even with such objects.

Wheatstone, in 1852, § 23, showed that the inversion of relief could also be obtained in the case of real objects. He used for the purpose a pair of reflecting prisms which he placed one before each eye and adjusted their inclination to one another so that, on looking at an object through them, the two images of the object were superposed. These prisms inverted the parallax, and they therefore presented to the eyes to some extent the appearances that would be produced if

* One such system was exhibited.

the relief of the object were reversed. The difficulty that many people experience in obtaining this pseudoscopic image is no doubt due to the fact that we are guided in our judgment of relief not only by the difference in parallax, but also by the differences in focus and in the light and shade. Also, obviously, many objects cannot be conceived as having reversed relief—for example, an object which is in

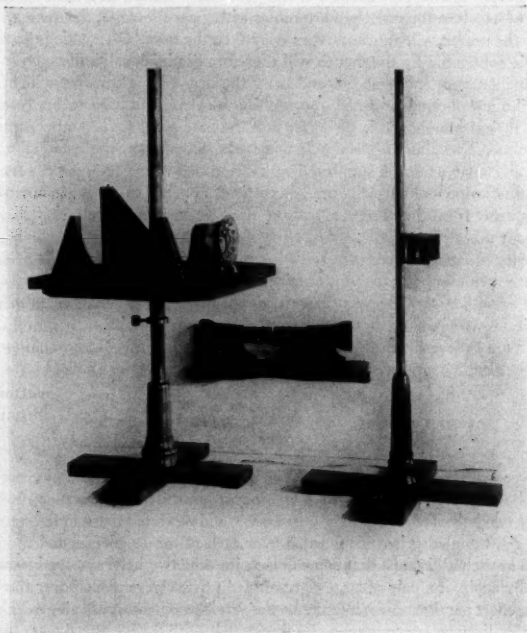


Fig. 4

front of a wall cannot readily be imagined to be behind that wall—so that only a limited number of objects can be seen pseudoscopically, even by those observers who can obtain the effect most readily.

An arrangement which Wheatstone devised for exhibiting the pseudoscopic effect* is shown in Fig. 4, which is a photograph of the apparatus at King's College.

* Wheatstone's original pseudoscope, as well as a later form made for him by Watkins & Hill, from the Court collection, were exhibited.

The right-hand stand carries the pair of prisms; the left-hand stand has a rotating table; a cup is mounted on one side of the table, and is still with the apparatus. On the near side is evidently the mount for a basin, and on the side opposite to the cup is apparently a support for a dish or plate.

A very curious effect is obtained when a wire cube, a glass beaker, or a tumbler, seen pseudoscopically, is rotated round a vertical axis*. If, for instance, the observer looks into the mouth of a tumbler with a pseudoscope, the mouth, though actually the nearer, will appear farther away than the base. On rotating the tumbler the angle subtended by the mouth will therefore diminish as its distance from the observer increases, and that subtended by the base of the tumbler will increase; but as the mouth appears to be approaching him and the base to be receding it *looks* as though the mouth is shrinking and the base swelling. In other words, the tumbler seems to alter in shape. If one looks down upon the rotating tumbler, which can conveniently be supported by a string tied round its middle, at an angle of 45° , the string (which of course is vertical) appears to be lying horizontally, pointing away from the observer, and the tumbler appears to be rotating over and over about this string as axis.

Wheatstone made a pair of equal wire cubes which he so placed in position in his mirror stereoscope that they combined apparently into one. These, by a slight rotation of either about a vertical axis, appear in *relief*, or flat, or *pseudoscopic*, depending upon whether the cubes are so placed that the images presented to the right and left eyes respectively are those that the cube itself would normally furnish, or are identical, or are reversed†. The original wire cubes are shown in Fig. 3.

OTHER STEREOSCOPIC APPARATUS IN THE WHEATSTONE COLLECTION AT KING'S COLLEGE

Between the two stands shown in Fig. 4 I have photographed a piece of apparatus—the forerunner of the Zeiss stereoscopic binocular. Wheatstone has separated the two halves of an ordinary opera glass, and mounted them in the same axial line. A right-angled prism is mounted over each of the eyepieces, and the bodies are fixed at such a distance that you can look into the two halves of the instrument. Then, obviously, if a pair of large right-angled prisms were placed over the objectives, to reflect parallel beams into the bodies, the instrument would give an enhanced interocular distance, corresponding approximately with the increased magnification.

Among the apparatus at King's College there is another exceedingly interesting piece of apparatus, being a sort of stereoscopic "wheel of life" (Fig. 5). A light cardboard drum, mounted on the rim of a wooden wheel, has twelve pairs of stereoscopic photographs of a model steam engine mounted at equal distances round its circumference. The successive photographs show the engine in slightly different positions, so that the series would show the engine in motion. The wooden wheel fits on an axis, and is attached to it by a pin. This axis is caused to rotate

* This experiment was demonstrated.

† A model illustrating this most interesting experiment was passed round the room.

intermittently by a rather interesting gear mechanism. There is a large fixed wheel, some 5 or 6 inches in diameter, and gearing to this is a wheel having one-twelfth as many teeth. A short link working on a pin, fixed to the face of this second wheel near its edge, is connected to an arm fixed to the axle on which the drum is mounted. The smaller wheel is rotated by a belt passing over a pulley, which is turned by a handle on the far side of the case. Thus the small wheel is carried round the fixed larger wheel, and the axle is moved intermittently by the cycloidal motion of the

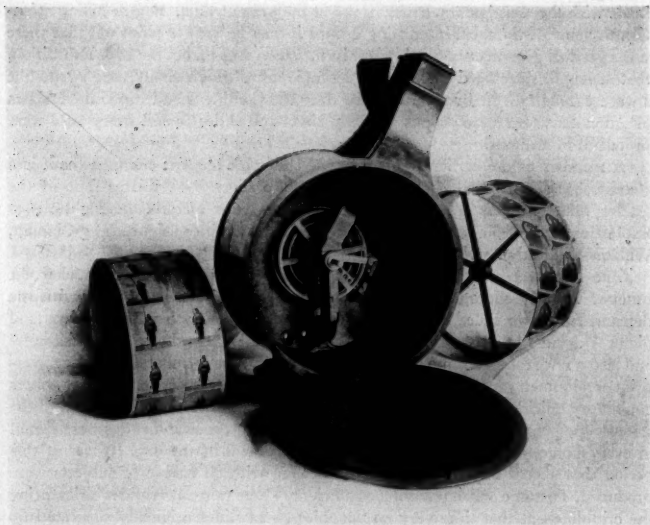


Fig. 5

pin to which the connecting link is attached. The observer looks in through a pair of prismatic lenses at the top of the rectangular projection seen on the upper right-hand part of the drum, and the light enters through an opening which can be closed by the shutter which is open in the figure. When the drum is in place in the box the circular lid in the front of the illustration is, of course, closed. A second set of photographs of a French soldier presenting arms is seen on the left of the case. The result, it must be confessed, is poor, as the time during which the drum is at rest is too short compared with the time taken for changing the picture, and also the drum is not sufficiently rigid to come properly to rest.

SLIDES

In the stereoscopes exhibited there is a carefully selected and representative collection of stereoscopic slides to show the various phases through which the stereoscope passed and the purposes to which it was put. In examining these, one must remember the dates at which they were produced, and not put them into competition with slides made with modern apparatus, and I do not therefore wish to suggest that these earlier productions can equal in excellence photographic slides made with the anastigmatic lenses in use at the present time. Before calling attention to some of the most interesting groups it may be well to point out that there was a gradual growth of public interest in the stereoscope from the 1851 Exhibition, that at length developed into the stereoscopic craze which about 1860 swept like a wave through Great Britain, and also over the Continent and the United States of America, when stereoscopes of many kinds and slides to suit every taste were produced in millions.

A number of the earlier Daguerreotype slides are shown: amongst them is a stereoscopic slide of Duboscq himself, two rare slides by Williams (one of the leading photographers in London at this time) of Queen Victoria opening the 1851 Exhibition, and also a selection of others (some beautifully coloured) by Claudet, Williams, and other early workers.

The next step was the introduction of collodion, discovered by Archer. A number of early slides, taken by this process, are shown. To comply with the demand for cheap slides, a series was produced (drawn by hand) of designs and pictures, generally in black and white. In this section are shown some examples and also a few from a unique set printed in colours. The introduction of the silver-print on albumen gave a great impetus to the lenticular stereoscope. Large numbers could very quickly and cheaply be produced, and the Exhibition of 1862 found the stereoscope practically at its height. Standard stereoscopes and slides were found in every home, and it was the fashionable instrument of the day. It was at this period that a very large number of sets dealing with all classes of subjects were produced. To name only a few will be sufficient. There were many slides caricaturing the crinoline—at that time very fashionable; ghost slides originally suggested by Sir David Brewster and others; polyscopic-stereoscopic slides, in which three complete slides were embodied in one slide; cosmoramic slides that could be used as opaque slides or as transparencies; and many others of which examples can be seen in the stereoscopes exhibited. Brewster pointed out that mother-of-pearl shells could be imitated by colouring the two pictures of the shell differently.

At this time, also, were produced a large number of stereoscopic views of London; many of these are now of great interest, as they show features of London of the first half of the nineteenth century no longer existing; taken as they generally were with the people and traffic, they show a London that is becoming unknown to the present generation. A fine series of the animals, birds, etc. at the Zoological Gardens, taken by special permission, was the first serious attempt to use the stereoscope as an aid to nature study; examples of these are shown. Then we have

a rare set of diagrams by the Stereoscopic Company to illustrate the problems of Euclid; and a unique set, specially produced by permission of Lady Franklin, of the relics of the Arctic Expedition of Sir John Franklin. In fact every phase of life was photographed and pressed into the service of this popular instrument.

Unfortunately, as it became popular, so at last it degenerated, and the more unscrupulous dealers produced slides of a questionable kind; these were condemned in the press and formed the subject of considerable correspondence. Judged by modern standard, the majority, although somewhat vulgar, would not now perhaps be considered either offensive or improper. However, this, together with the sale of slides which were only composed of two exactly similar pictures, caused the stereoscope to go out of fashion, as the gentry and better classes of the community ceased to take an interest in it, and by about 1868 the craze had entirely died out. Since this time there have been many attempts to revive an interest in this beautiful instrument, but with only a very limited success. The United Stereoscopic Society was the last and most successful effort in this direction, and a number of beautiful slides taken by members of that Society can be examined this evening, and amply prove that in the hands of one who is both an artist and an expert photographer few scientific instruments can give such magnificent results. The slides shown were not made for sale to the public, but were in the private collection of Dr Walsh Owen, its President; on his death nearly all his slides were purchased by Mr Court and added to his collection.

STEREOSCOPIC PROJECTION

Many attempts have been made to project stereoscopic views and cinematograph pictures, so as to make them visible to a whole audience. Any such projection must arrange for the one picture to be seen by the right eye of every observer, and the other picture to be seen by the left eye. This separation has been effected in four ways:

A. By the use of complementary colours—usually red and green—both for projecting and viewing, as in the well-known “anaglyphs.”

B. By the use of polarised light. The pictures are projected by two beams of light from two lanterns, these beams being polarised in planes at right angles to one another. Each member of the audience is furnished with a pair of spectacles containing “analysers,” which pass only the beam polarised in their own plane. The screen must be a matt-silvered one, as the ordinary white screen depolarises the light. A pair of projecting polarisers and a number of analysers are exhibited (Court collection).

C. By the use of intermittent light and viewers. Each of the audience looks at the screen through a pair of apertures which are open alternately, the aperture through which either eye looks being open at the same time as the corresponding picture is being projected.

D. The pictures are projected side by side, and each of the audience is provided with a pair of spectacles which superposes the pictures. Professor E. A. Bost

of Toronto University used a liquid prism of variable angle for this purpose. They have also been superposed by a double reflection; a mirror is mounted on a spectacle frame in front of, say, the right eye, at 45° to the line of sight, reflecting the light to the right; just beyond this is a second mirror, approximately parallel to the first mirror, to reflect the light back into its path. But this second mirror is hinged, so that its angle can be varied by means of a screw, which the wearer adjusts for himself, until the two images appear to be superposed. It is obvious that the angle will have to be varied according to the distance that the wearer is away from the screen—the nearer he is the greater will be the angle between the mirrors.

All four of these methods have their obvious disadvantages and consequently have never become popular.

Before concluding I must thank the Delegacy of King's College for the loan of the original Wheatstone apparatus, and also for allowing me to take the photographs which are here reproduced. I desire also to thank Sir Henry Lyons for lending me so many of the stereoscopes from the Science Museum. I have to thank the Editor of the *Illustrated London News* for kindly giving me the anaglyph viewers; also the Royal Photographic Society for the loan of two magazine stereoscopes, Miss Violet Jeffery for a third one, and Professor von Rohr for a number of references. Last, but not least, I wish to thank Mr Thos. H. Court for the loan of stereoscopes, for all the beautiful slides which I have been able to exhibit, and for much valuable assistance.

THE PRIMORDIAL COEFFICIENTS OF ASYMMETRICAL LENSES

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ABSTRACT. An easily calculable system of sixteen magnitudes is constructed for the representation of the properties of asymmetrical lenses by the addition of four lengths to the magnitudes previously used. The equations connecting this system and the coefficients of the eikonal and of the characteristic function, the equations for combining systems or moving their reference points, and the identities between the coefficients are expressed in matrix form. It is shown that the eleven variables present in a system of three separated astigmatic lenses only yield nine degrees of freedom, and such a system cannot, if the reference points are placed where the outer lenses meet the axis, represent the general system, which has ten degrees of freedom.

The elementary theory of asymmetrical lens systems has been carried in two recent papers* to a stage which suffices for many practical requirements, but further development is obviously necessary to attain a state comparable with that reached by Gauss in the theory of the simpler class of symmetrical instruments. The object of the present paper is to bring the theory of the asymmetrical instrument to a stage corresponding to that already attained for the symmetrical system.

On account of the large number of variables as well as of equations involved when we are dealing with the asymmetrical instrument, it is important to present in the most concise form the relations which may be established. For this reason matrices have been employed. As the elementary properties of matrices needed for the discussion will be unknown to some readers a section dealing with them has been included in this paper. Any slight disadvantage the matrix method of presentation may have for such readers is more than compensated by the greater ease of remembering the relations when expressed in so symmetrical a form.

* "On Toric Lenses," *Trans. Opt. Soc.* 29 1927-8 (71). "Canonical Forms in the Theory of Asymmetrical Optical Systems," *Trans. Opt. Soc.* 29 1927-8 (88). For brevity these papers will be referred to as T.L. and C.F., numbers in brackets following these letters denoting the equations so numbered in the paper mentioned.

The Elementary Theory of Matrices

The matrices we shall have to consider are those known as square matrices, owing to the equality between the number of rows lying across the paper and columns in the perpendicular direction. The set of n^2 quantities

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

forms a matrix of order n which is written in full above but for brevity may be denoted by (a) . Each of the n^2 quantities may be chosen arbitrarily, and (a) simply denotes the set arranged in the above definite order as determined by the distinguishing suffixes. It is important to notice that a change in the first of the two suffixes denotes a change of row, and a change in the second a change of column. In a similar way we may write down another matrix (b) , also of the n th order, in which the element placed where the h th row (the top row being the first) meets the k th column (the first being on the extreme left) is b_{hk} .

The matrices (a) and (b) are merely the two groups of numbers with each number in a definite position; no mathematical processes have yet been carried out on these. We now define the n th order matrix (c) as the product of the matrices (a) and (b) , a relation which we express in either the condensed form

$$(a)(b) = (c)$$

or the extended form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix},$$

when all the c 's are calculated from the equation

$$c_{pq} = \sum_{k=1}^n a_{pk} b_{kq} \quad \dots (1).$$

Thus any particular c involves only the a 's of the corresponding row and the b 's of the corresponding column. It will be observed that the formula (1) is identical with that by which we write down a determinant which is equal to the product of two given determinants, so that we can at once say that

$$|a| |b| = |c|.$$

This equation however merely expresses a single relation, while (1) or the equivalent matrix equation

$$(a)(b) = (c)$$

expresses n^2 relations.

In a similar way we may define (d) as the product of (b) and (a) , or

$$(b)(a) = (d),$$

where $d_{p,q} = \sum_{k=1}^n b_{pk} a_{kq}$. It is important to observe that (c) and (d) are quite different matrices, i.e. the order in which the factors are arranged affects the product. The reader can easily verify that the two products are distinct by taking a simple numerical example, e.g.

$$\begin{pmatrix} 1 & 4 & 6 \\ -1 & 7 & -3 \\ 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & -6 & 0 \\ 1 & 1 & -1 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 24 & 22 & 26 \\ -4 & 1 & -22 \\ 19 & 8 & 25 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -6 & 0 \\ 1 & 1 & -1 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 & 6 \\ -1 & 7 & -3 \\ 2 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 8 & -34 & 30 \\ -2 & 11 & -2 \\ 9 & 40 & 31 \end{pmatrix}.$$

The two determinants $|c|$ and $|d|$ corresponding to the two product matrices are of course equal; thus the determinantal relations corresponding to the above matrix products are

$$(-53) \times 66 = -3498$$

and

$$66 \times (-53) = -3498.$$

When three matrices are multiplied together, say

$$(a)(b)(e) = (g)$$

the elements of (g) are to be determined from

$$g_{pq} = \sum_{h=1}^n \sum_{k=1}^n a_{ph} b_{hk} e_{kq} \quad \dots\dots(2).$$

Now suppose that $(a)(b) = (c)$ and $(b)(e) = (f)$, so that by (1)

$$c_{pk} = \sum_{h=1}^n a_{ph} b_{hk}$$

and

$$f_{hq} = \sum_{k=1}^n b_{hk} e_{kq}.$$

Equation (2) then gives

$$g_{pq} = \sum_{k=1}^n c_{pk} e_{kq} = \sum_{h=1}^n a_{ph} f_{hq}$$

or

$$(g) = (a)(b)(e) = (c)(e) = (a)(f),$$

that is to say in forming a continued product of matrices we may group successive matrices for multiplication as is convenient, always supposing that we do not disturb the order of the factors.

If every element of the matrix contains a factor k we adopt a convention which enables us to write k as a multiplying factor outside the matrix. Thus

$$\begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{n1} & ka_{n2} & \dots & ka_{nn} \end{pmatrix} = k \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

In a matrix product such factors are obviously commutative.

In general we assign no value to a matrix, but we may do so in two cases—the zero matrix and the unit matrix. From (1) it is obvious that if all the elements of one factor matrix are zero the product matrix will have all its elements also zero. In accordance with the factor convention we have adopted we may regard such a matrix as one whose elements contain zero as a common factor and accordingly assign the value zero also to the matrix itself.

We call a matrix (a) a unit matrix if all the elements a_{hh} which have equal suffixes (and are therefore situated on the diagonal joining the top left and the bottom right corners) are equal to unity, and all the remaining elements, which are characterised by unequal suffixes, are zero. From (1) it follows that if (a) is a unit matrix $c_{pq} = b_{pq}$, or (c) = (b), and similarly if (b) is a unit matrix, $c_{pq} = a_{pq}$ or (c) = (a). We are therefore justified in assigning the value unity to any unit matrix.

If we add up the suffixes in the typical terms of (1) or (2), corresponding to the sum $p + q$ on the left we obtain $p + q + 2k$ or $p + q + 2h + 2k$ on the right. As h and k are both integral, we always obtain an even or odd sum for every term on the right according as the sum on the left is even or odd. It follows that in any matrix equation we may alter the signs of all the elements occupying positions where the sum of the suffixes is odd without affecting the truth of the relation.

From (1) and (2) it may be seen that we may reverse the order of all the matrices in a product if we interchange rows and columns in the factors and also in the product. Thus from the numerical example already given we have

$$\begin{pmatrix} 1 & -1 & 2 \\ 4 & 7 & 0 \\ 6 & -3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ -6 & 1 & 4 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 8 & -2 & 9 \\ -34 & 11 & 40 \\ 30 & -2 & 31 \end{pmatrix}.$$

Gaussian Relations in Matrix Form

Before considering the asymmetrical instrument it may be helpful to put the well-known relations for the symmetrical instrument in matrix form in order to exhibit the suitability of the matrix calculus for optical purposes.

The four Gaussian coefficients a, b, c, d may be defined in a variety of ways. If the subject is approached through the one-one correspondence between object and image points, the fundamental relation is

$$axx' + bx' - cx - d = 0.$$

If we take the path length with ray coordinates as our basis we define the coefficients by means of the eikonal equation

$$a\mathcal{E} = \text{constant} + \frac{1}{2}b(M^2 + N^2) - (MM' + NN') + \frac{1}{2}c(M'^2 + N'^2),$$

and if we utilise point coordinates instead of ray coordinates the equation for the characteristic function

$$dV = \text{constant} - \frac{1}{2}c(y^2 + z^2) + (yy' + zz') - \frac{1}{2}b(y'^2 + z'^2)$$

will be regarded as fundamental. Whichever method we adopt we add the condition

$$\begin{vmatrix} b & d \\ a & c \end{vmatrix} = 1,$$

which clearly implies that b and c are pure numbers, d a length, and a a curvature. The four well-known geometrical equations giving the point in terms of the ray coordinates are then represented by the single matrix equation

$$\begin{pmatrix} b & 1 \\ 1 & c \end{pmatrix} \begin{pmatrix} M & N \\ -M' & -N' \end{pmatrix} = a \begin{pmatrix} y & z \\ y' & z' \end{pmatrix},$$

and the converse relations by

$$\begin{pmatrix} b & 1 \\ 1 & c \end{pmatrix} \begin{pmatrix} -y' & -z' \\ y & z \end{pmatrix} = d \begin{pmatrix} M' & N' \\ M & N \end{pmatrix}.$$

Again, if we combine two systems 1 and 2 to yield the compound system 12, the equations are

$$\begin{pmatrix} b_1 & d_1 \\ a_1 & c_1 \end{pmatrix} \begin{pmatrix} b_2 & d_2 \\ a_2 & c_2 \end{pmatrix} = \begin{pmatrix} b_{12} & d_{12} \\ a_{12} & c_{12} \end{pmatrix}.$$

If the origins are moved through reduced distances x and x' we have only to write

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & d \\ a & c \end{pmatrix} \begin{pmatrix} 1 & -x' \\ 0 & 1 \end{pmatrix},$$

instead of the central matrix alone, the difference in sign of x and x' being due to the adoption of the same instead of opposite directions along the axis as positive in both object and image spaces.

Lest it should be thought that the presentation of these familiar relations in matrix form is artificial and pointless, it will be as well to say that though many different methods have been employed very extensively for the numerical computation of the Gaussian coefficients, there is no doubt that one particular method is at once the least laborious and the most accurate. This is precisely the method of step by step multiplication of matrices. The data are usually presented in the form of the powers of the individual surfaces and their reduced axial separations. If the powers are $\kappa_1, \kappa_2, \kappa_3 \dots$ and the reduced separations $\tau_1, \tau_2, \tau_3 \dots$ the best equations to use in computation, whether written in matrix form or not, are

$$\begin{pmatrix} 1 & 0 \\ \kappa_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tau_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tau_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa_3 & 1 \end{pmatrix} \dots = \begin{pmatrix} b & d \\ a & c \end{pmatrix},$$

and the product is usually found by working through the system from one end to the other either in the order 1, 2, 3 ... or in the reverse order.

All the well-known results in special types of system follow at a glance from the matrix formula. For instance, a thin system is obtained by putting all the τ 's equal to zero in the above product, so that the alternate matrices become unit matrices, and obviously by (1)

$$\begin{pmatrix} 1 & 0 \\ \kappa_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa_3 & 1 \end{pmatrix} \dots = \begin{pmatrix} 1 & 0 \\ \kappa_1 + \kappa_2 + \kappa_3 + \dots & 1 \end{pmatrix}.$$

It will be seen in the subsequent parts of this paper that there is a close analogy between these Gaussian relations and the corresponding equations for the more complex asymmetrical system, an analogy which the matrix notation exhibits admirably.

Old Asymmetrical Relations in Matrix Form

The convenience of the matrix notation in discussing the more complex problems of the asymmetrical instrument may be illustrated by some of the results already established. The eikonal, apart from an additive constant, contains ten coefficients $a, b, c \dots j^*$, the defining equation being

$$\begin{aligned} \mathcal{E} = & \frac{1}{2}aM^2 + \frac{1}{2}bN^2 + \frac{1}{2}cM'^2 + \frac{1}{2}dN'^2 \\ & + eMN + fM'N' - gMN' - hM'N - iMM' - jNN' \dots (3). \end{aligned}$$

These coefficients are not calculated directly, it being preferable to compute twelve functions which are defined in T.L. (9). These twelve equations are all contained in the single matrix equation

$$\begin{pmatrix} j & g & 0 & 0 \\ h & i & 0 & 0 \\ 0 & 0 & j & h \\ 0 & 0 & g & i \end{pmatrix} \begin{pmatrix} a & -e & i & -g \\ -e & b & -h & j \\ 1 & 0 & c & -f \\ 0 & 1 & -f & d \end{pmatrix} = |ij| \begin{pmatrix} \mathcal{A} & \mathcal{I} & 1 & 0 \\ \mathcal{I} & \mathcal{B} & 0 & 1 \\ \mathcal{I} & \mathcal{H} & \mathcal{C} & \mathcal{U} \\ \mathcal{I} & \mathcal{I} & \mathcal{V} & \mathcal{D} \end{pmatrix} \dots (4),$$

which in addition shows explicitly that by $|ij|$ we are to understand $ij - gh$. Not only can we express so many equations in the form of a single relation, but we can also solve them all as readily as we could solve for a single variable in another notation. For from the definitions of $\mathcal{I}, \mathcal{H}, \mathcal{C}, \mathcal{U}$, we note that on dividing equation (4) throughout by $|ij|$ it becomes

$$\begin{pmatrix} \mathcal{I} & \mathcal{I} & 0 & 0 \\ \mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I} & \mathcal{H} \\ 0 & 0 & \mathcal{I} & \mathcal{I} \end{pmatrix} \begin{pmatrix} a & -e & i & -g \\ -e & b & -h & j \\ 1 & 0 & c & -f \\ 0 & 1 & -f & d \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{I} & 1 & 0 \\ \mathcal{I} & \mathcal{B} & 0 & 1 \\ \mathcal{I} & \mathcal{H} & \mathcal{C} & \mathcal{U} \\ \mathcal{I} & \mathcal{I} & \mathcal{V} & \mathcal{D} \end{pmatrix} \dots (5).$$

* Lower case italics, indicating lengths, will normally be used for the coefficients of the eikonal.

Now the product

$$\begin{pmatrix} \mathcal{F} & -\mathcal{G} & 0 & 0 \\ -\mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{F} & -\mathcal{H} \\ 0 & 0 & -\mathcal{G} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{F} & \mathcal{G} & 0 & 0 \\ \mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{F} & \mathcal{H} \\ 0 & 0 & \mathcal{G} & \mathcal{I} \end{pmatrix}$$

by (1) is the product of a unit matrix and the factor $\mathcal{F}\mathcal{I} - \mathcal{G}\mathcal{H}$; we therefore have the solution

$$\begin{pmatrix} \mathcal{F} & -\mathcal{G} & 0 & 0 \\ -\mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{F} & -\mathcal{H} \\ 0 & 0 & -\mathcal{G} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{F} & 1 & 0 \\ \mathcal{I} & \mathcal{B} & 0 & 1 \\ \mathcal{F} & \mathcal{H} & \mathcal{C} & \mathcal{U} \\ \mathcal{G} & \mathcal{I} & \mathcal{V} & \mathcal{D} \end{pmatrix} = |\mathcal{F}\mathcal{I}| \begin{pmatrix} a & -e & i & -g \\ -e & b & -h & j \\ 1 & 0 & c & -f \\ 0 & 1 & -f & d \end{pmatrix},$$

or

$$\begin{pmatrix} \mathcal{F} & \mathcal{G} & 0 & 0 \\ \mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{F} & \mathcal{H} \\ 0 & 0 & \mathcal{G} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{A} & -\mathcal{F} & 1 & 0 \\ -\mathcal{I} & \mathcal{B} & 0 & 1 \\ \mathcal{F} & -\mathcal{H} & \mathcal{C} & -\mathcal{U} \\ -\mathcal{G} & \mathcal{I} & \mathcal{V} & \mathcal{D} \end{pmatrix} = |\mathcal{F}\mathcal{I}| \begin{pmatrix} a & e & i & g \\ e & b & h & j \\ 1 & 0 & c & f \\ 0 & 1 & f & d \end{pmatrix} \dots\dots(6),$$

which contains all the results previously given in the twelve equations T.L. (10).

Introduction of Linear Coefficients

In the Gaussian theory of symmetrical instruments four directly calculated coefficients are considered, of which one is a power, two are pure numbers, and one is a length. Of the twelve directly calculated coefficients (represented by script letters) for the asymmetrical instrument four are powers and the other eight are pure numbers. It appears likely that a convenient complete system will involve as many coefficients representing lengths as there are coefficients representing powers. We therefore consider what four lengths would be suitable for incorporation in the system. None of the lengths represented by italic letters is suitable, and in fact the only possible choice appears to involve the minors of g, h, i, j in the determinant

$$\begin{vmatrix} a & e & i & g \\ e & b & h & j \\ i & h & c & f \\ g & j & f & d \end{vmatrix} = |w| \text{ say,}$$

which has frequently appeared in these investigations. These minors are of dimensions (length)³, and we obtain lengths by dividing by $|ij|$, exactly as in selecting the twelve original functions. We thus have the definitions in duplicate

$$(w) \begin{pmatrix} \mathcal{C} & -\mathcal{U} & -1 & 0 \\ -\mathcal{V} & \mathcal{D} & 0 & -1 \\ -1 & 0 & \mathcal{A} & -\mathcal{F} \\ 0 & -1 & -\mathcal{I} & \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{C} & -\mathcal{C} & 0 & 0 \\ -\mathcal{P} & \mathcal{R} & 0 & 0 \\ 0 & 0 & \mathcal{Q} & -\mathcal{P} \\ 0 & 0 & -\mathcal{Q} & \mathcal{R} \end{pmatrix} \dots\dots(7),$$

a form which may be compared with a slight modification of equation (5), viz.

$$\begin{pmatrix} \mathcal{F} & -\mathcal{S} & 0 & 0 \\ -\mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I} & -\mathcal{H} \\ 0 & 0 & -\mathcal{S} & \mathcal{I} \end{pmatrix} (w) = \begin{pmatrix} \mathcal{A} & -\mathcal{I} & 1 & 0 \\ -\mathcal{I} & \mathcal{B} & 0 & 1 \\ 1 & 0 & \mathcal{C} & -\mathcal{U} \\ 0 & 1 & -\mathcal{Q} & \mathcal{D} \end{pmatrix} \dots (8).$$

We may regard (7) and (8) as the standard forms in which to express the relations between the script variables and the coefficients of the eikonal.

The Coefficients of the Characteristic Function

It is important to know the relations between the script variables and the coefficients of the characteristic function. This function, apart from a constant term, is defined as

$$V = \frac{1}{2}Ay^2 + \frac{1}{2}Bx^2 + \frac{1}{2}Cy'^2 + \frac{1}{2}Dz'^2 + Eyz + Fy'z' - Gyz' - Hy'z - Iyy' - Jzz' \dots (9)*,$$

and by C.F. (9) the coefficients are connected with those of the eikonal by

$$\begin{pmatrix} a & e & i & g \\ e & b & h & j \\ i & h & c & f \\ g & j & f & d \end{pmatrix} \begin{pmatrix} A & E & I & G \\ E & B & H & J \\ I & H & C & F \\ G & J & F & D \end{pmatrix} = -1 \dots (10),$$

or more briefly $(w)(W) = -1$.

Owing to the symmetry of the two component matrices, by (1) the product of these two matrices is not affected by changing the order of the factors. We therefore obtain at once from (7) and (8) for the standard relations we are seeking

$$(W) \begin{pmatrix} \mathcal{Q} & -\mathcal{P} & 0 & 0 \\ -\mathcal{P} & \mathcal{R} & 0 & 0 \\ 0 & 0 & \mathcal{Q} & -\mathcal{P} \\ 0 & 0 & -\mathcal{P} & \mathcal{R} \end{pmatrix} = - \begin{pmatrix} \mathcal{C} & -\mathcal{U} & -1 & 0 \\ -\mathcal{Q} & \mathcal{D} & 0 & -1 \\ -1 & 0 & \mathcal{A} & -\mathcal{I} \\ 0 & -1 & -\mathcal{I} & \mathcal{B} \end{pmatrix} \dots (11)$$

and $\begin{pmatrix} \mathcal{A} & -\mathcal{I} & 1 & 0 \\ -\mathcal{I} & \mathcal{B} & 0 & 1 \\ 1 & 0 & \mathcal{C} & -\mathcal{U} \\ 0 & 1 & -\mathcal{Q} & \mathcal{D} \end{pmatrix} (W) = - \begin{pmatrix} \mathcal{I} & -\mathcal{I} & 0 & 0 \\ -\mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I} & -\mathcal{H} \\ 0 & 0 & -\mathcal{S} & \mathcal{I} \end{pmatrix} \dots (12).$

Equation (11) is immediately convertible into

$$\begin{pmatrix} \mathcal{R} & \mathcal{P} & 0 & 0 \\ \mathcal{P} & \mathcal{Q} & 0 & 0 \\ 0 & 0 & \mathcal{R} & \mathcal{Q} \\ 0 & 0 & \mathcal{P} & \mathcal{Q} \end{pmatrix} \begin{pmatrix} \mathcal{C} & -\mathcal{U} & -1 & 0 \\ -\mathcal{U} & \mathcal{D} & 0 & -1 \\ \mathcal{Q} & -\mathcal{P} & \mathcal{A} & -\mathcal{I} \\ -\mathcal{P} & \mathcal{R} & -\mathcal{I} & \mathcal{B} \end{pmatrix} = -|\mathcal{Q}\mathcal{R}| \begin{pmatrix} A & E & I & G \\ E & B & H & J \\ -1 & 0 & C & F \\ 0 & -1 & F & D \end{pmatrix} \dots (13),$$

* Italic capitals, denoting curvatures, will normally be used for the coefficients of the characteristic function.

the equation which will normally be used to find the coefficients of the characteristic function, or into

$$\begin{pmatrix} J & G & 0 & 0 \\ H & I & 0 & 0 \\ 0 & 0 & J & H \\ 0 & 0 & G & I \end{pmatrix} \begin{pmatrix} A & -E & I & -G \\ -E & B & -H & J \\ -I & 0 & C & -F \\ 0 & -I & -F & D \end{pmatrix} = -|IJ| \begin{pmatrix} \mathcal{C} & \mathcal{V} & -I & 0 \\ \mathcal{U} & \mathcal{D} & 0 & -I \\ \mathcal{Q} & \mathcal{O} & \mathcal{A} & \mathcal{I} \\ \mathcal{P} & \mathcal{R} & \mathcal{I} & \mathcal{B} \end{pmatrix} \quad \text{.....(14),}$$

which is convenient for the evaluation of script coefficients when those of the characteristic function are given.

The Identities

Since the system of sixteen script coefficients which we have now obtained is derived from a set of ten quantities, there must be six independent relations connecting them. We have two relations not involving $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ from the agreement of the pairs of values of e and f given in (6), and similarly two relations independent of $\mathcal{I}$, $\mathcal{H}$, $\mathcal{J}$, $\mathcal{I}$ from the duplicate expressions for E and F contained in (13). All the relations which exist may be found at once from a pair of equations such as (7) and (8). For the elimination of (w) gives

$$\begin{pmatrix} \mathcal{A} & \mathcal{I} & I & 0 \\ \mathcal{I} & \mathcal{B} & 0 & I \\ I & 0 & \mathcal{C} & \mathcal{U} \\ 0 & I & \mathcal{V} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathcal{C} & \mathcal{U} & -I & 0 \\ \mathcal{V} & \mathcal{D} & \mathcal{O} & -I \\ -I & 0 & \mathcal{A} & \mathcal{I} \\ 0 & -I & \mathcal{I} & \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathcal{I} & 0 & 0 \\ \mathcal{H} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I} & \mathcal{H} \\ 0 & 0 & \mathcal{I} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{Q} & \mathcal{O} & 0 & 0 \\ \mathcal{P} & \mathcal{R} & 0 & 0 \\ 0 & 0 & \mathcal{Q} & \mathcal{P} \\ 0 & 0 & \mathcal{O} & \mathcal{R} \end{pmatrix} \quad \text{.....(15)*,}$$

which formally represents equalities between pure numbers. From (15) and the further identities mentioned we see that all the relations are included in the equations

$$(\mathcal{W})(\mathcal{W}') = (\mathcal{W}')(\mathcal{W}) = I \quad \text{.....(16),}$$

where $(\mathcal{W}) = \begin{pmatrix} \mathcal{A} & \mathcal{I} & \mathcal{Q} & \mathcal{O} \\ \mathcal{I} & \mathcal{B} & \mathcal{P} & \mathcal{R} \\ \mathcal{I} & \mathcal{H} & \mathcal{C} & \mathcal{U} \\ \mathcal{I} & \mathcal{I} & \mathcal{V} & \mathcal{D} \end{pmatrix}$, $(\mathcal{W}') = \begin{pmatrix} \mathcal{C} & \mathcal{V} & -\mathcal{Q} & -\mathcal{P} \\ \mathcal{U} & \mathcal{D} & -\mathcal{O} & -\mathcal{R} \\ -\mathcal{I} & -\mathcal{I} & \mathcal{A} & \mathcal{I} \\ -\mathcal{H} & -\mathcal{I} & \mathcal{I} & \mathcal{B} \end{pmatrix}$... (17).

Since $|\mathcal{W}| = |\mathcal{W}'|$, and the determinant is obviously positive for a thin system, it follows that $|\mathcal{W}| = I$. Many other identities can be constructed, but they may all be reduced to one of equations (16).

* From (6) and (13) we note that the conditions for the existence of the eikonal and of the characteristic function are $|\mathcal{I}\mathcal{I}| \neq 0$ and $|\mathcal{Q}\mathcal{R}| \neq 0$ respectively. Equation (15) shows that both exist if

$$\begin{vmatrix} \mathcal{A} & \mathcal{I} & I & 0 \\ \mathcal{I} & \mathcal{B} & 0 & I \\ I & 0 & \mathcal{C} & \mathcal{U} \\ 0 & I & \mathcal{V} & \mathcal{D} \end{vmatrix} \neq 0.$$

The Combination of Two Systems

The equations for constructing the eikonal coefficients of a combination of two systems in terms of those of its parts are given in T.L. (2). It is a simple matter to transform these into the script variables, when it is found that they may be written in the simple form

$$(\mathcal{U}_1)(\mathcal{U}_2) = (\mathcal{U}_{12}) \quad \text{.....(18),}$$

where $(\mathcal{U}_1)$, $(\mathcal{U}_2)$, $(\mathcal{U}_{12})$ are the matrices defined in (17) for the first portion of the system, the second portion, and the whole respectively. It is assumed that the internal reference points for the two portions coincide, and that the complete system is referred to the external reference points for the two portions.

If the reference points for any system are moved outwards through reduced distances τ and τ' respectively in the initial and final media, $(\mathcal{U})$ is replaced by

$$\begin{pmatrix} 1 & 0 & -\tau & 0 \\ 0 & 1 & 0 & -\tau \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\mathcal{U}) \begin{pmatrix} 1 & 0 & -\tau' & 0 \\ 0 & 1 & 0 & -\tau' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{.....(19).}$$

For a single surface or for a thin lens, with the reference points where the surface meets the axis,

$$(\mathcal{U}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \psi & \pi & 1 & 0 \\ \pi & \phi & 0 & 1 \end{pmatrix} \quad \text{.....(20)}$$

in a notation employed in T.L. From the last three equations it is seen that $\mathcal{S}$, $\mathcal{H}$, $\mathcal{J}$, $\mathcal{J}$ are unaltered by movements of the reference points, and $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ by the introduction of additional refractions at the reference points.

The fundamental importance of $(\mathcal{U})$ makes it desirable to remember how the letters are arranged. The positions of the four quarterly groups involve no difficulty. On the leading diagonal the letters are in their natural order, and on the other diagonal the corner positions are taken by the letters which occur earliest in the alphabet, the next letters taking the inside positions. The remaining letters are so arranged as to correspond to clockwise movement about the intersection of the diagonals. The changes in the order in the equations connecting these quantities with the coefficients of the eikonal and the characteristic function should be particularly noted.

Geometrical Equations

Assuming that omitted lines and columns (here all but the first) in a matrix consist entirely of zeroes, equations (3) and (9) may be written

$$(M \quad N \quad -M' \quad -N') (w) \begin{pmatrix} M \\ N \\ -M' \\ -N' \end{pmatrix} = 2\mathcal{E}$$

and
$$(y \quad z \quad -y' \quad -z') (W) \begin{pmatrix} y \\ z \\ -y' \\ -z' \end{pmatrix} = 2V,$$

the product matrices being considered to have only a single element. From the familiar properties of $\mathcal{E}$ and V we can write down the geometrical relations

$$(M \quad N \quad -M' \quad -N') (w) = (y \quad z \quad y' \quad z') \quad \dots\dots(21)$$

and
$$(-y \quad -z \quad y' \quad z') (W) = (M \quad N \quad M' \quad N') \quad \dots\dots(22),$$

the last of the equations needed to complete the analogy with the symmetrical system.

Rotation of Axes

As a final example we may consider the alterations in the coefficients brought about by rotating the axes of y and z and of y' and z' about the axes of x and x' respectively. For brevity we denote the matrix

$$\begin{pmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \beta & \sin \beta \\ 0 & 0 & -\sin \beta & \cos \beta \end{pmatrix}$$

by (α, β) , the bracketed quantities being the angles α and β . After rotating the object space axes through the angle α and the image space axes through the angle β the matrices (w) , (W) and $(\mathcal{W})$ are respectively replaced by

$$(\alpha, \beta) (w) (-\alpha, -\beta),$$

$$(\alpha, \beta) (W) (-\alpha, -\beta),$$

and
$$(-\alpha, -\alpha) (\mathcal{W}) (\beta, \beta).$$

The first of these results was found in c.f., and the other two follow immediately from the standard equations connecting the different systems of coefficients, together with

$$(\alpha, \beta) (\gamma, \delta) = (\alpha + \gamma, \beta + \delta)$$

and
$$(0, 0) = I.$$

The suitability of $(\mathcal{U})$ as a system for initial computation is indicated by the segregation of α and β in distinct matrices respectively preceding and following $(\mathcal{U})$: conversely the presence of these two angles in mixed matrices on either side of (w) and (W) suggests that these coefficients are not advantageously computed by direct formulae.

System of Three Separated Thin Lenses

In T.L. and C.F. the need for at least three separated thin lenses in a model representing the general system has been considered from two distinct points of view. Whether three such lenses actually suffice has not yet been discussed. Two thin separated lenses enable us to realise all the degrees of freedom of any symmetrical system for any choice of reference points, except that one of the Gaussian coefficients is necessarily negative if the reference points coincide with the lenses. In the asymmetrical problem three separated lenses are insufficient if the positions of the reference points are arbitrary. For from (18), (19), and (20) if the points are placed where the outer lenses meet the axis we see that $\mathcal{C} = \mathcal{P}$, and hence by (13) $G = H$. With this exception it is obvious that the three lenses may be chosen to yield arbitrarily selected values for the coefficients. It follows that the eleven constructional elements of a system of three separated lenses only enable nine arbitrary conditions to be satisfied.

THE THEORY OF APLANATIC SURFACES

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ABSTRACT. The necessary and sufficient condition that an optical system should have a pair of aplanatic surfaces is that the eikonal of the system can be expressed as a homogeneous function of the first order of three variables, each of which is a linear function of the direction cosines of the ray in the object and image spaces. Methods are given for finding the equations of these surfaces when the eikonal is given, and for finding the eikonal when the surfaces are given. Conjugate aplanatic surfaces are similar apart from arbitrary uniform unidirectional extensions or compressions.

In general only one pair of aplanatic surfaces is possible, but in spherically symmetrical systems two pairs are found. In addition there may be aplanatic imagery at isolated points.

OUR knowledge of the circumstances in which strict point to point correspondence subsists between a surface and its image, as determined by the criterion of geometrical optics, is surprisingly limited. The point to point correspondence between two planes has naturally received much attention, since it underlies most of the work which has been done on systems of lenses. Apart from the trivial case in which the object and image coincide with a single reflecting or refracting surface, the only other example which has been widely recognised is the pair of aplanatic surfaces associated with refraction at a single spherical surface. All these examples have the properties that the conjugate surfaces are similar and that the magnification over the whole surface is constant. It is not unimportant to know whether these properties are generally true and to determine what other surfaces can be imaged without aberration.

We can avoid the violation of any optical laws by basing our investigation on the properties of the characteristic function or of the eikonal. The only additional assumption we shall make is that the function we employ (in this investigation the eikonal) is an arbitrary function of its variables. In effect this implies that by the use of a sufficient number of refracting surfaces, suitably spaced and shaped, and with suitable intermediate media, we may attain complete control over all the aberrations of the system. The justification for this assumption is the absence of any evidence which would lead us to doubt its truth.

We choose orthogonal axes arbitrarily in both object and image spaces. The linear coordinates will be denoted by x, y, z and x', y', z' respectively, and the corresponding direction cosines by L, M, N and L', M', N' , of which M, N, M', N' will be regarded in the first place as the independent variables. For simplicity we omit refractive indices. If the external media differ, we may understand either that

x, y, z, x', y', z' represent "optical" coordinates, or that L, M, N, L', M', N' denote the product of the direction cosines and the refractive index; in the latter event the identities between the direction cosines yield

$$L^2 + M^2 + N^2 = \mu^2, \quad L'^2 + M'^2 + N'^2 = \mu'^2,$$

where μ and μ' are the refractive indices of the media occupying the object space and the image space respectively.

Let us now consider two surfaces, one in the object space and one in the image space, of which the cartesian equations are

$$f(x, y, z) = 0 \quad \text{and} \quad f'(x', y', z') = 0.$$

In order that these should be conjugate aplanatic surfaces, it is necessary and sufficient that corresponding to any x, y, z which satisfies the former we should be able to determine unique values of x', y', z' which satisfy the latter. That is to say, the latter surface will be derived from the former by substitutions which we may represent by

$$x' = X'(x, y, z),$$

$$y' = Y'(x, y, z),$$

$$z' = Z'(x, y, z),$$

where X', Y', Z' are single-valued functions of x, y, z . Evidently the solution of these equations will be

$$x = X(x', y', z'),$$

$$y = Y(x', y', z'),$$

$$z = Z(x', y', z'),$$

where X, Y, Z are also single-valued functions of their variables. We may express these conditions in a more symmetrical form by supposing x, y, z and x', y', z' to be functions of three other variables p, q, r . Thus we may write

$$\left. \begin{aligned} x &= \xi(p, q, r), & x' &= \xi'(p, q, r) \\ y &= \eta(p, q, r), & y' &= \eta'(p, q, r) \\ z &= \zeta(p, q, r), & z' &= \zeta'(p, q, r) \end{aligned} \right\} \dots\dots(1),$$

equations soluble in the form

$$\left. \begin{aligned} p &= P(x, y, z) = P'(x', y', z') \\ q &= Q(x, y, z) = Q'(x', y', z') \\ r &= R(x, y, z) = R'(x', y', z') \end{aligned} \right\} \dots\dots(2).$$

The equations (1) ensure that corresponding to given values of p, q, r we obtain one definite point (x, y, z) in the object space and another definite point (x', y', z') in the image space. From any relation

$$\theta(p, q, r) = 0 \quad \dots\dots(3)$$

connecting the three variables p, q, r we obtain, by the substitutions (2), one equation connecting x, y, z and another equation connecting x', y', z' , and these are

obviously the equations of surfaces. Since these surfaces correspond point for point, we shall obtain aplanatic imagery over extended surfaces by finding a relation (3) between any three variables which satisfy the conditions (1) and (2).

These considerations lead us to expect that where aplanatic imagery is possible the eikonal can be expressed as a function of three variables. We will therefore assume that the eikonal $\mathcal{E}$, apart from terms of the form

$$\text{constant} + lL + mM + nN + l'L' + m'M' + n'N',$$

which have no significance beyond a change of origin and so can be disregarded, is a function of three variables α, β, γ . Remembering that the point coordinates are necessarily determined by differentiating the eikonal, we naturally seek a solution by forming a relation between the differential coefficients of $\mathcal{E}$ with respect to α, β, γ .

Let ϵ be any function of α, β, γ . Then these quantities are connected by an equation

$$\phi_1(\alpha, \beta, \gamma, \epsilon) = 0,$$

which in general is not homogeneous. Let the dummy variable δ be introduced to make the equation homogeneous, so that it becomes

$$\phi(\alpha, \beta, \gamma, \delta, \epsilon) = 0 \quad \dots\dots(4).$$

This equation may be differentiated with respect to $\alpha, \beta, \gamma, \delta, \epsilon$ in turn, and the five new equations, as a consequence of homogeneity, are, like equation (4) itself, essentially equations in the four variables $\alpha/\epsilon, \beta/\epsilon, \gamma/\epsilon, \delta/\epsilon$. By eliminating these four variables between the six equations, we can form two independent equations

$$\left. \begin{aligned} \psi_1(\phi_\alpha, \phi_\beta, \phi_\gamma, \phi_\delta, \phi_\epsilon) &= 0 \\ \psi_2(\phi_\alpha, \phi_\beta, \phi_\gamma, \phi_\delta, \phi_\epsilon) &= 0 \end{aligned} \right\} \quad \dots\dots(5),$$

neither of which is necessarily homogeneous. If one of them is homogeneous, it is the equation we are seeking. Otherwise we can form a homogeneous equation, for the two relations (5) are the conditions that the two equations

$$\left. \begin{aligned} \psi_1(t\phi_\alpha, t\phi_\beta, t\phi_\gamma, t\phi_\delta, t\phi_\epsilon) &= 0 \\ \psi_2(t\phi_\alpha, t\phi_\beta, t\phi_\gamma, t\phi_\delta, t\phi_\epsilon) &= 0 \end{aligned} \right\} \quad \dots\dots(6)$$

have a common root, viz. $t = 1$. Thus the eliminant of equations (6) regarded as functions of t vanishes, that is a homogeneous equation

$$\theta(\phi_\alpha, \phi_\beta, \phi_\gamma, \phi_\delta, \phi_\epsilon) = 0 \quad \dots\dots(7)$$

has been found.

Now since ϵ is a function of the three independent variables α, β, γ , we have by (4), on varying each in turn,

$$\frac{\phi_\alpha}{\epsilon_\alpha} = \frac{\phi_\beta}{\epsilon_\beta} = \frac{\phi_\gamma}{\epsilon_\gamma} = -\phi_\epsilon,$$

or since (7) is homogeneous we may write

$$\theta \left(\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma, -\frac{\phi_\delta}{\phi}, -\frac{1}{I} \right) = 0 \quad \dots\dots(8).$$

From this expression we see that an equation between the differential coefficients of ϵ is obtained if, and only if, $\phi_\delta = 0$, that is to say if ϕ is a homogeneous function of $\alpha, \beta, \gamma, \epsilon$; in other words ϵ is a homogeneous function of α, β, γ of order one.

Now by one of the fundamental properties of the eikonal the incident and emergent portions of the ray L, M, N, L', M', N' pass through

$$\left(0, \frac{d\mathcal{E}}{dM}, \frac{d\mathcal{E}}{dN} \right) \text{ and } \left(0, -\frac{d\mathcal{E}}{dM'}, -\frac{d\mathcal{E}}{dN'} \right)$$

respectively, that is through

$$\left(0, \frac{\partial \mathcal{E}}{\partial M} - \frac{M \partial \mathcal{E}}{L \partial L}, \frac{\partial \mathcal{E}}{\partial N} - \frac{N \partial \mathcal{E}}{L \partial L} \right) \text{ and } \left(0, -\frac{\partial \mathcal{E}}{\partial M'} + \frac{M' \partial \mathcal{E}}{L' \partial L'}, -\frac{\partial \mathcal{E}}{\partial N'} + \frac{N' \partial \mathcal{E}}{L' \partial L'} \right).$$

Thus whatever values ρ and ρ' may have, the points

$$\left(\frac{\partial \mathcal{E}}{\partial L} + \rho L, \frac{\partial \mathcal{E}}{\partial M} + \rho M, \frac{\partial \mathcal{E}}{\partial N} + \rho N \right) \text{ and } \left(-\frac{\partial \mathcal{E}}{\partial L'} + \rho' L', -\frac{\partial \mathcal{E}}{\partial M'} + \rho' M', -\frac{\partial \mathcal{E}}{\partial N'} + \rho' N' \right)$$

lie one on each external part of the ray. We therefore obtain solutions by supposing that $\mathcal{E}$ is a function of ϵ and therefore depends essentially on three functions α, β, γ of the direction cosines, these functions being so chosen that

$$\frac{\partial \mathcal{E}}{\partial L} + \rho L, \frac{\partial \mathcal{E}}{\partial M} + \rho M, \frac{\partial \mathcal{E}}{\partial N} + \rho N, -\frac{\partial \mathcal{E}}{\partial L'} + \rho' L', -\frac{\partial \mathcal{E}}{\partial M'} + \rho' M', -\frac{\partial \mathcal{E}}{\partial N'} + \rho' N'$$

can be expressed in terms of $\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma$ only. The equations

$$\left. \begin{aligned} x &= \rho L + \frac{d\mathcal{E}}{d\epsilon} (\alpha_L \epsilon_\alpha + \beta_L \epsilon_\beta + \gamma_L \epsilon_\gamma) \\ y &= \rho M + \frac{d\mathcal{E}}{d\epsilon} (\alpha_M \epsilon_\alpha + \beta_M \epsilon_\beta + \gamma_M \epsilon_\gamma) \\ z &= \rho N + \frac{d\mathcal{E}}{d\epsilon} (\alpha_N \epsilon_\alpha + \beta_N \epsilon_\beta + \gamma_N \epsilon_\gamma) \\ x' &= \rho' L' - \frac{d\mathcal{E}}{d\epsilon} (\alpha_{L'} \epsilon_\alpha + \beta_{L'} \epsilon_\beta + \gamma_{L'} \epsilon_\gamma) \\ y' &= \rho' M' - \frac{d\mathcal{E}}{d\epsilon} (\alpha_{M'} \epsilon_\alpha + \beta_{M'} \epsilon_\beta + \gamma_{M'} \epsilon_\gamma) \\ z' &= \rho' N' - \frac{d\mathcal{E}}{d\epsilon} (\alpha_{N'} \epsilon_\alpha + \beta_{N'} \epsilon_\beta + \gamma_{N'} \epsilon_\gamma) \end{aligned} \right\} \quad \dots\dots(9)$$

must therefore be free from L, M, N, L', M', N' . All the conditions are obviously satisfied by putting

$$\left. \begin{aligned} \rho &= \rho' = \mathcal{E} - \epsilon = 0 \\ \alpha &= a_0 + a_1 L + a_2 M + a_3 N - a_4 L' - a_5 M' - a_6 N' \\ \beta &= b_0 + b_1 L + b_2 M + b_3 N - b_4 L' - b_5 M' - b_6 N' \\ \gamma &= c_0 + c_1 L + c_2 M + c_3 N - c_4 L' - c_5 M' - c_6 N' \end{aligned} \right\} \quad \dots\dots(10),$$

where all the a 's, b 's, and c 's are constants. The equations for x , y , and z are then

$$\left. \begin{aligned} x &= a_1 \mathcal{E}_a + b_1 \mathcal{E}_\beta + c_1 \mathcal{E}_\gamma \\ y &= a_2 \mathcal{E}_a + b_2 \mathcal{E}_\beta + c_2 \mathcal{E}_\gamma \\ z &= a_3 \mathcal{E}_a + b_3 \mathcal{E}_\beta + c_3 \mathcal{E}_\gamma \\ x' &= a_4 \mathcal{E}_a + b_4 \mathcal{E}_\beta + c_4 \mathcal{E}_\gamma \\ y' &= a_5 \mathcal{E}_a + b_5 \mathcal{E}_\beta + c_5 \mathcal{E}_\gamma \\ z' &= a_6 \mathcal{E}_a + b_6 \mathcal{E}_\beta + c_6 \mathcal{E}_\gamma \end{aligned} \right\} \dots\dots(11),$$

of which the solutions are

$$\left. \begin{aligned} \frac{\mathcal{E}_a}{|xb_2c_3|} &= \frac{\mathcal{E}_\beta}{|a_1yc_3|} = \frac{\mathcal{E}_\gamma}{|a_1b_2x|} = \frac{1}{|a_1b_2c_3|} \\ \frac{\mathcal{E}_a}{|x'b_5c_6|} &= \frac{\mathcal{E}_\beta}{|a_4y'c_6|} = \frac{\mathcal{E}_\gamma}{|a_4b_5x'|} = \frac{1}{|a_4b_5c_6|} \end{aligned} \right\} \dots\dots(12),$$

and

where the determinants are identified by the elements of their leading diagonals.

It should be noted that finite values of ρ and ρ' are equivalent to the presence in α , β , γ of terms in $(L^2 + M^2 + N^2)$ and $(L'^2 + M'^2 + N'^2)$. There is thus no loss of generality involved in supposing $\rho = \rho' = 0$, and the solution which has been given is evidently the general solution of the problem. Since the optical cosine law must be obeyed for any elementary displacement along the aplanatic surface, we might have expected to be led to such expressions as (10).

The coordinates of the image point corresponding to a given object point on the aplanatic surface are obtained by substituting for $\mathcal{E}_a$, $\mathcal{E}_\beta$, $\mathcal{E}_\gamma$ from the first part of (12) in the last three of equations (11). Similarly the object point is given in terms of the image point by eliminating $\mathcal{E}_a$, $\mathcal{E}_\beta$, $\mathcal{E}_\gamma$ from the first three of equations (11) using the second half of (12). These equations show that the aplanatic image of a surface has the form obtainable by subjecting the object surface to a series of compressions or extensions, each arbitrary but uniform in magnitude and arbitrary but definite in direction. Thus the aplanatic image of a plane is necessarily a plane: that of a closed conic is also a closed conic: the image of a paraboloid is also a paraboloid and that of a hyperboloid is a hyperboloid.

There is a sense in which we may say that in the aplanatic imagery of surfaces distortion is impossible, in spite of the fact that the conditions imposed only concern the stigmatic character of the reunion of rays in the image space.

The Converse Problem

The position now reached is that we know that a system represented by an eikonal which is a homogeneous function of the first order of three variables, each linear in the direction cosines of a ray, has a pair of aplanatic surfaces, and we have shown how to find those surfaces when the eikonal is given. The converse problem of finding the eikonal having given aplanatic surfaces can also be solved. For since

the solution has the same form irrespective of directional compressions and extensions, we may without loss of generality suppose the object and image surfaces to be identical in shape and size. The problem is then identical with that of finding the eikonal for a given refracting surface, the solution of which is known*. The problem of finding the refracting surface corresponding to a given eikonal†, it will be noticed, is similarly identical with the trivial case (in which the object and image surfaces are coincident) of this investigation. It follows that the process of finding the eikonal which secures aplanatic imagery over given surfaces is essentially identical with that of finding the surfaces free from aberration with a given eikonal. Illustrations of this process will be found in the earlier work to which reference has been made.

Axially Symmetric Systems

The variables in axially symmetric systems are necessarily much more limited than those given in (10). The planes of coordinates may be chosen so that y' depends only on y and z' only on z ; moreover these two relations must be exactly similar. The equations (10) thus take the form

$$\alpha = aL + a'L' + a'',$$

$$\beta = bM + b'M' + b'',$$

$$\gamma = bN + b'N' + b'',$$

and by considering opposite sides of any axial section it is seen that $b'' = 0$.

These simplifications in effect mean that in these systems the three degrees of freedom are reduced to two; the two independent functions of $\mathcal{E}$ entering into the equation of the aplanatic surfaces are, in the notation we have hitherto employed, $\mathcal{E}_a$ and $(\mathcal{E}_\rho^2 + \mathcal{E}_\gamma^2)^{\frac{1}{2}}$.

The Number of Aplanatic Surfaces

We have adopted as a general solution a form in which $\rho = \rho' = 0$: this assumption requires us, in any attempt to make an exhaustive examination into the aplanatic surfaces of a system, to take into account the different forms in which the variables may be expressed without altering the eikonal itself. Where no alternative forms exist there is only one pair of aplanatic surfaces. That more than one pair may exist is seen by taking a suitable example. Let $\mathcal{E} = (\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}$, where

$$\alpha = pL + qL',$$

$$\beta = pM + qM',$$

$$\gamma = pN + qN'.$$

Since the direction cosines satisfy the conditions

$$L^2 + M^2 + N^2 = 1 = L'^2 + M'^2 + N'^2,$$

* *Trans. Opt. Soc.* 28 1926-7 (265).

† *Trans. Opt. Soc.* 28 1926-7 (274).

we can, without altering $\mathcal{E}$, take as variables

$$\alpha' = qL + pL',$$

$$\beta' = qM + pM',$$

$$\gamma' = qN + pN',$$

in terms of which we have $\mathcal{E} = (\alpha'^2 + \beta'^2 + \gamma'^2)^{\frac{1}{2}}$. If the external media have refractive indices μ and μ' , from the first form of $\mathcal{E}$ we obtain the pair of aplanatic surfaces

$$\mu^2 (x^2 + y^2 + z^2) = p^2,$$

$$\mu'^2 (x'^2 + y'^2 + z'^2) = q^2,$$

and from the second form the pair

$$\mu^2 (x^2 + y^2 + z^2) = q^2,$$

$$\mu'^2 (x'^2 + y'^2 + z'^2) = p^2.$$

Any real change of effective variable such as is here illustrated will necessarily depend upon the invariant relation satisfied by direction cosines, and in no other example than that afforded by a system having spherical symmetry is it obvious that a complete change of variable can be effected. It thus appears that in general there will only be one pair of aplanatic surfaces, but in systems having spherical symmetry two pairs will be found. It must be added that in addition to these surfaces isolated points may exist at which the image is aplanatic. For example, at the centre of curvature of any set of concentric refracting spherical surfaces there is no aberration, and the sine condition is satisfied no matter what the direction of the normal to the object surface may be.

The Independence of Aberrations

The form of the solution to this problem forces us—unless an alternative solution has been overlooked—to one conclusion which was not expected. We started with the assumption that we have complete control over all the aberrations, but the solution we have found allows no such liberty. For example, we might have expected to find it possible for distortion to be present in an image surface without affecting the stigmatic character of the imagery. It would appear that any trace of distortion, or any curvature in the image of a plane object, is inevitably accompanied by failure to achieve perfect reunion of the image rays at some part of the field.

At first sight this conclusion seems inconsistent with our ability to construct an eikonal for the aplanatic imagery of a surface of arbitrary form on an arbitrary scale. Let us suppose the ratio of image to object is infinitely great for one system and zero for another. The finite arbitrary surface in the former system is conjugate to the surface at infinity, and the surface at infinity is also conjugate to the finite

image of a surface of any other arbitrary shape in the second system. It would appear that by combining the two we could secure aplanatic imagery of the first surface on the second. There is, however, a fallacy in the argument, as is readily seen on attempting to construct the eikonal for zero or infinitely great values of the magnification. The form of the aplanatic image surface corresponding to the surface at infinity is not arbitrary.

THE ERRORS OF A REFLECTING PRISM

BY INSTRUCTOR CAPTAIN T. Y. BAKER, F.INST.P., R.N.

MS. received, 6th February, 1928. Read, 14th June, 1928.

ABSTRACT. A prism with two reflecting faces designed to fulfil a particular purpose in an optical instrument gives rise to errors due to (a) inaccuracy of manufacture and (b) inaccuracy in mounting the prism in the instrument. The effects of these errors are investigated.

The special case of the constant deviation prism, in which the three parts of the ray inside the prism nominally lie in a plane, gives rise to an error of the second order, the value of which is obtained.

INTRODUCTION

A PRISM designed to fulfil a specific purpose, and worked out in detail as regards its dimensions, especially in respect of the angles between the faces, can never be manufactured rigidly according to specification. Notably it is not possible to attain perfection in all the angles nor is it possible to set the prism up in the instrument for which it is intended in the exact manner that the designer desired. As a result the rays of light which traverse it emerge in directions slightly different from those aimed at and the instrument of which the prism forms part contains resultant errors which must either be neutralised by some adjustable mechanism, optical or mechanical, or, alternatively, the instrument must be calibrated and the errors allowed for. In this paper an attempt is made to deal with the errors in the direction of an emergent ray in the case of a prism which contains two reflecting surfaces and two transmission faces across which the axial ray is supposed to pass normally. In practice the angle between the reflecting faces will be slightly different from that specified, and the axial ray, partly owing to inaccurate manufacture and partly owing to inaccurate setting up, will traverse the transmission faces at small angles with the normals so that an error of direction results from the refraction effects.

If the prism is traversed by pencils of parallel light the only resulting errors are those of direction of the emergent rays; if, on the other hand, the light is convergent then the shift in the image position depends not only on the error in the direction of the emergent axial ray but also upon the position of the point of emergence. In what follows the former case only is considered and the prism and the light which traverses it may be dealt with as in a previous paper* by means of a spherical diagram.

* T. Y. Baker, *Trans. Opt. Soc.* 29 1927-8 (49).

THE IDEAL DIAGRAM

In the most general case that can be considered the prism has two plane faces set at some angle, $180^\circ - \phi$, with one another and light is reflected internally from them in succession. Previously to the first reflection any ray must have entered the prism from the outside and in some specified direction relatively to the rest of the instrument. The designer intends, in general, that the ray shall enter the prism normally to the entrance face and leave the prism normally to the emergence face. There is thus no refraction effect on the ray either on the entrance or emergence and there are three definite directions in space occupied by the three parts of that ray. The ray itself consists of parts of three intersecting straight lines. In the most general case the three parts are not coplanar and the prism is a skew prism, but if

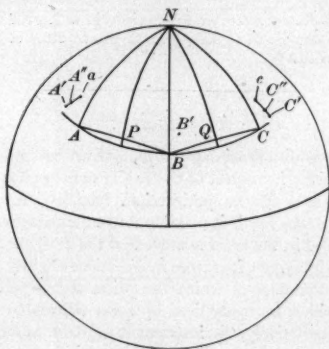


Fig. 1

the prism is designed so that they do lie in a plane we get the more limited case of a constant deviation prism. Fig. 1 gives the spherical representation of the general case. NP and NQ are two great circles on the sphere, inclined at an angle ϕ with one another, and parallel to the reflecting faces of the prism. The point A represents the direction of the entrant ray, that is to say, the direction, and sense, of that ray is parallel to the outward radius of the sphere drawn to the point A . This point A also represents the inward normal to the entrant face of the prism.

After reflection in the face NP the ray travels in a direction represented by B where the great circle AB is bisected at right angles by the great circle NP . The second reflection gives the emergent ray the direction represented by C where BC is at right angles to NQ and $BQ = QC$.

We see from this graphical construction that the arcs NA , NB , NC are all equal ($= \theta$) and for convenience of reference and description we can treat N as

if it were the pole of the earth so that points may be specified in terms of their latitude and longitude, for example, A , B , and C all have the same latitude $90^\circ - \theta$.

In the previous paper it was shown that the design of a prism with two reflecting faces always leaves one degree of freedom. The two great circles NP and NQ may be moved about upon the surface of the sphere provided N remains fixed, and the angle between the circles remains fixed. Under these conditions A and C will remain stationary although B will move about.

Usually the designer will arrange that the ray is incident upon the two surfaces at equal angles in which case AP , PB , BQ , QC are all equal as also are the angles ANP , BNP , BNQ , CNQ . This will be taken to be customary practice in this paper.

To sum up, we have, in the ideal case contemplated by the designer, that the entrant ray A passes into the prism without refraction; it strikes the first reflecting surface at an inclination $AP (= p)$ and the second reflecting face at the same inclination $BQ (= p)$ and emerges along the normal C to the emergence face. The total change in direction of the ray is from A to C . Further in the figure

$$AN = BN = CN = \theta,$$

$$ANP = BNP = BNQ = CNQ = \frac{1}{2}\phi.$$

If the great circle AC be drawn it will be at right angles to BN . If the arc $AC = 2\psi$ it can be shown that

$$\sin \psi = \sin \theta \sin \phi \quad \dots\dots(1),$$

$$\sin p = \sin \theta \sin \frac{1}{2}\phi \quad \dots\dots(2).$$

The figure thus represents the general case of the skew prism as the designer would like it to be made and set up. It becomes a constant deviation prism when the three points A , B , and C lie upon a great circle which can be the case only if $\theta = 90^\circ$ with the result that $\psi = \phi = 2p$.

THE ACTUAL DIAGRAM

In practice, owing to errors of manufacture and setting up, the entrant ray is not represented by A but by some point A' near A . At the same time the normal to the first transmission face is represented not by A but by an adjacent point a . Both the points A' and a will lie within a few minutes of arc from A . When the ray enters the prism it is refracted. It lies in the plane containing the incident ray and the normal so that the spherical representation of the refracted ray is a point A'' upon the great circle joining A' to a . The law of refraction gives

$$\sin A'a = n \sin A''a.$$

Within the limits of accuracy dealt with, this is equivalent to

$$A'a = n.A''a.$$

Now the whole of the relevant area round A , namely that which includes the points A' , A'' , and a , is small and for all practical purposes may be treated as a plane area.

Let the differences in "latitude" between A and A' , A'' , a be $\epsilon_1, \epsilon_2, \epsilon_3$ and the differences in "longitude" $\lambda_1, \lambda_2, \lambda_3$. Treating the area as plane we obtain

$$\epsilon_2 = \{\epsilon_1 + \epsilon_3 (n-1)\}/n$$

and

$$\lambda_2 = \{\lambda_1 + \lambda_3 (n-1)\}/n.$$

The point A'' has now to be reflected in the great circle NP and the reflected ray B'' has coordinates ϵ_2, λ_2 relatively to B (see Fig. 1).

Let the emergent ray and the normal to the emergent face be C' and c , the ray just before the emergence being C'' . The three points $A', A'',$ and a have been taken to be to the west and north of A ; $C', C'',$ and c to the east and north of C and in dealing with the algebra connecting the ϵ 's and the λ 's equations must be put down in proper relationship to the figure. The corresponding ϵ 's and λ 's for $C', C'',$ and c are $\epsilon_1', \epsilon_2', \epsilon_3'$ and $\lambda_1', \lambda_2', \lambda_3'$. We have

$$\epsilon_2' = \{\epsilon_1' + \epsilon_3' (n-1)\}/n$$

and

$$\lambda_2' = \{\lambda_1' + \lambda_3' (n-1)\}/n.$$

Reflecting the ray C'' back again in the surface NQ we must make the reflected ray agree with the reflection of A'' in NP . This gives rise to

$$\epsilon_1 + \epsilon_3 (n-1) = \epsilon_1' + \epsilon_3' (n-1)$$

and

$$\lambda_1 + \lambda_3 (n-1) + \lambda_1' + \lambda_3' (n-1) = 0$$

or

$$(\epsilon_1 - \epsilon_1') = (n-1) (\epsilon_3' - \epsilon_3) \quad \dots\dots(3)$$

and

$$\lambda_1 + \lambda_1' = -(n-1) (\lambda_3 + \lambda_3') \quad \dots\dots(4).$$

The total displacement of the ray is from A' to C' and we obtain by spherical trigonometry

$$\begin{aligned} \cos A'C' &= \cos NA' \cos NC' + \sin NA' \sin NC' \cos (2\phi + \lambda_1 + \lambda_1') \\ &= \cos (\theta - \epsilon_1) \cos (\theta - \epsilon_1') + \sin (\theta - \epsilon_1) \sin (\theta - \epsilon_1') \cos (2\phi + \lambda_1 + \lambda_1'). \end{aligned}$$

If we put $A'C' = 2\psi + 2\Delta\psi$ the above equation reduces to

$$-2\Delta\psi \cos \psi = \cos \theta \sin \phi (\epsilon_1 + \epsilon_1') - \sin \theta \cos \phi (\lambda_1 + \lambda_1') \quad \dots\dots(5)$$

provided we retain only small quantities of the first order.

Equations (3) and (4) enable us to express ϵ_1' and λ_1' in terms of $\epsilon_3, \lambda_3; \epsilon_3', \lambda_3'; \epsilon_1, \lambda_1$, i.e. in terms of the inaccuracy of manufacture of the prism and inaccuracy of the setting up. It will be seen that the resulting error in the angular displacement of the ray is of the first order in the general case of the skew prism.

THE CONSTANT DEVIATION PRISM

If in formula (5) we put $\theta = 90^\circ$ so as to deal with the special case in which, for the ideal prism, all three parts of the ray are coplanar and perpendicular to the line of intersection of the two reflecting faces then we have seen that $\psi = \phi$ and we obtain

$$\begin{aligned} 2\Delta\psi &= \lambda_1 + \lambda_1' \\ &= -(n-1) \{\lambda_3 + \lambda_3'\} \text{ from (4).} \end{aligned}$$

This is a first order variation from the ideal case but the error is one due to inaccuracy of manufacture, not to inaccuracy of setting up, so that the total angular displacement aimed at is not 2ϕ but $2\phi - (n-1)(\lambda_3 + \lambda_3')$. It is a constant angle of the prism and is independent, to first order, of the manner in which the prism is set up in the instrument and prisms used in this manner, for example the pentagonals of a rangefinder, do deflect the light through a constant angle as far as concerns the first order values of the errors of setting the prisms in the instrument.

In practice, in many cases, that first order accuracy is insufficient. The deflecting angle of a rangefinder pentagonal needs to be constant to a fraction of a second and it is not sufficient to consider that it is constant without investigating the errors of a higher order than the first which are due to maladjustment.

THE SECOND ORDER ERRORS OF A CONSTANT DEVIATION PRISM

In Fig. 2, NP , NQ represent the two reflecting faces at an angle $180^\circ - \phi$ with one another, A the ideal incident ray, and C the ideal emergent ray;

$$AP = PB = BQ = QC = \frac{1}{2}\phi.$$

Actually the entrant ray is at A' and the emergent ray at C' and a and c are the normals to the two transmission faces. The corresponding rays on the glass

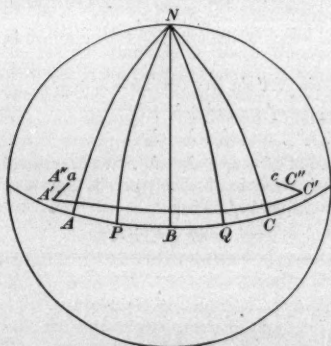


Fig. 2

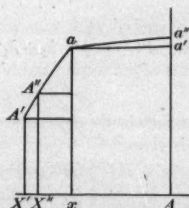


Fig. 3

are A' and C' . The spherical area near A is drawn on a larger scale in Fig. 3. $A'A''a$ is a great circle; $A'X'$, $A''X''$, ax are also great circles, viz. meridians drawn from the point N . Consider first the point a . Its latitude and longitude coordinates relatively to the point A are ϵ_3 , λ_3 . Draw aa' a parallel of latitude through a and aa'' a great circle through a at right angles to the meridian of A . We have

$$aa' = \lambda_3 \cos ax = \xi_3 \cos \epsilon_3 = \lambda_3 (1 - \frac{1}{2}\epsilon_3^2).$$

Since λ_3 is a small quantity of the first order it is clear that in putting $aa' = \lambda_3$ we are neglecting a quantity of the third order, viz. $\frac{1}{2}\lambda_3\epsilon_3^2$, so that if we aim at retaining only quantities of the second order in the measurement of the arc $A'C'$ the approximation is a legitimate one.

Again from the right-angled spherical triangle Naa''

$$\cos Na = \cos aa'' \cos Na''$$

$$\sin \epsilon_3 = \cos aa'' \sin Aa''$$

or

giving

$$(\epsilon_3 - \frac{1}{2}\epsilon_3^2) = \{1 - \frac{1}{2}(aa'')^2\} \{Aa'' - \frac{1}{2}(Aa'')^2\}$$

showing that to second order approximation

$$Aa'' = \epsilon_3$$

which is equivalent to stating that $a'a''$ is a quantity of the third order. Also the refraction equation

$$\sin A'a = n \sin A''a$$

reduces to

$$A'a - \frac{1}{2}(A'a)^2 = n \{A''a - \frac{1}{2}(A''a)^2\}$$

or, once more, $A'a = n.A''a$ is a legitimate approximation.

Without going into all the details of the trigonometry it can be stated that the whole of the area of Fig. 3 being of linear dimensions of the first order the treatment of it as a plane area involves the neglect of third order terms only and we are therefore justified in using as before equations (3) and (4) as legitimate approximations to connect the coordinates of A'' , C'' with those of A' , a , C' , c .

The exact equation from which the length of the arc $A'C'$ is determined is, as before,

$$\begin{aligned} \cos A'C' &= \cos NA' \cos NC' + \sin NA' \sin NC' \cos (2\phi + \lambda_1 + \lambda_1') \\ &= \sin \epsilon_1 \sin \epsilon_1' + \cos \epsilon_1 \cos \epsilon_1' \cos (2\phi + \lambda_1 + \lambda_1') \\ &= \epsilon_1 \epsilon_1' + \{1 - \frac{1}{2}(\epsilon_1^2 + \epsilon_1'^2)\} \cos (2\phi + \lambda_1 + \lambda_1'). \end{aligned}$$

If ϵ_1 and ϵ_1' were zero we should have $A'C' = 2\phi + \lambda_1 + \lambda_1'$. We can therefore write $A'C' = 2\phi + \lambda_1 + \lambda_1' + \delta$ and it is clear that δ could then be expressed in terms of the second and higher orders of ϵ_1 and ϵ_1' . Hence

$$\cos A'C' = \cos (2\phi + \lambda_1 + \lambda_1') - \delta \sin (\phi + \lambda_1 + \lambda_1')$$

is a sufficiently close approximation since δ is itself of second order dimensions. This gives

$$\begin{aligned} \delta &= \frac{1}{2}(\epsilon_1 + \epsilon_1')^2 \cot (2\phi + \lambda_1 + \lambda_1') - \epsilon_1 \epsilon_1' \operatorname{cosec} (2\phi + \lambda_1 + \lambda_1') \\ &= \frac{1}{2}(\epsilon_1 + \epsilon_1')^2 \cot 2\phi' - \epsilon_1 \epsilon_1' \operatorname{cosec} 2\phi', \end{aligned}$$

writing $2\phi'$ for $2\phi + \lambda_1 + \lambda_1'$. Substituting in the formula for δ the value of ϵ_1' derived from (3), we obtain, after reduction,

$$\delta = \frac{1}{4}(n-1)^2(\epsilon_3' - \epsilon_3)^2 \cot \phi' - \{\epsilon_1 - \frac{1}{2}(n-1)(\epsilon_3' - \epsilon_3)\}^2 \tan \phi'.$$

Hence

$$\begin{aligned} A'C' &= 2\phi + \lambda_1 + \lambda_1' + \frac{1}{4}(n-1)^2(\epsilon_3' - \epsilon_3)^2 \cot \phi' - \{\epsilon_1 - \frac{1}{2}(n-1)(\epsilon_3' - \epsilon_3)\}^2 \tan \phi' \\ &= 2\phi - (n-1)(\lambda_2 + \lambda_3') + \frac{1}{4}(n-1)^2(\epsilon_3' - \epsilon_3)^2 \cot \phi \\ &\quad - \{\epsilon_1 - \frac{1}{2}(n-1)(\epsilon_3' - \epsilon_3)\}^2 \tan \phi \end{aligned}$$

in which the substitution of ϕ for ϕ' introduces an error of third order only. Thus it will be seen that $A'C'$ is not constant to second order but varies in respect of the last term. $A'C'$ is, however, a maximum when $\epsilon_1 = \frac{1}{2}(n-1)(\epsilon'_3 - \epsilon_3)$ and for any position of setting up we can write

$$A'C' = \text{constant} - y^2 \tan \phi$$

where y stands for $\epsilon_1 - \frac{1}{2}(n-1)(\epsilon'_3 - \epsilon_3)$.

Ordinarily one would anticipate the manufacturing error $\epsilon'_3 - \epsilon_3$ to be not more than a few minutes. If we put 5 minutes as an outside value we find that the coefficient of the term $\frac{1}{2}(n-1)^2(\epsilon'_3 - \epsilon_3)^2 \cot \phi$ is less than a thirtieth of a second and is in practice negligible.

AN EXAMPLE FROM THE RANGEFINDER PENTAGONAL

The formula above can be utilised to evaluate the errors due to bad setting up of the pentagonal of a rangefinder. The prism is not perfectly made and the best manner to set it up is that in which the value of $A'C'$ is the maximum. This entails $\epsilon_1 = -\epsilon'_1 = \frac{1}{2}(n-1)(\epsilon'_3 - \epsilon_3)$. Consequently in Fig. 2, C' would be as far below the "equator" as A' is above it and the great circle $A'C'$ would pass through B . The point N which represents the intersection of the two plane faces is not at right angles to the plane $A'C'$, i.e. to the plane of the rangefinding triangle, but is at right angles to the line B in that plane.

In Fig. 4 is shown the end pentagonal of the rangefinder. Let A' , C' be supposed to be in the plane of the paper and the prism be set up to begin with so that the two reflecting faces are also in the plane of the paper. When in this position the inward normal G_1 at the entrance face points upwards from the plane of the paper by an amount ϵ_3 , the outward normal to the emergence face upwards by an amount ϵ'_3 . In order to set the prism so that the maximum angle of deflection is being used, it must be tilted round the pecked line B by an amount

$$0.707(n-1)(\epsilon'_3 - \epsilon_3)$$

the sense being that which gives a right-hand screw motion when combined with the direction of B . This gives the best position for the prism. Any departure from this may alter the angle between the entrant and emergent rays.

Now any angular displacement from this best position can be resolved into a small rotation x round B' , a small rotation y round B , and a small rotation z round the vertical. Of these three quantities only y will affect the angle between A' and C' , the magnitude of the effect being $y^2 \tan \phi$, i.e. for the pentagonal y^2 .

If we wish to keep the consequent error below 0.2 second the value of y must not exceed about $3\frac{1}{2}$ minutes of arc.

The tilt x of the prism round B does not affect the angle between A' and C' but does affect the direction of the emergent light. It is this tilt which is required for the correction of halving error.

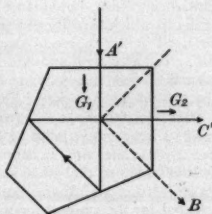


Fig. 4

REVIEW

The Legibility of Print (Special Report Series No. 110 of the Medical Research Council). By R. L. PYKE, M.A. Pp. 123; 1 Fig. and 1 Plate. (London: H.M. Stationery Office, 1926.) Price 4s. net.

This report gives an extensive summary of previous work on the legibility of type and an account of recent investigations carried out in the Psychological Laboratories of Cambridge University under the control of a committee of the Medical Research Council. In reviewing past work the author criticises various methods which have been adopted by other investigators and discusses various criteria which may be employed to arrange a number of specimens of type in order of legibility. The meaning of legibility is considered at length. In the experimental work here described various tests were applied. The author confines his attention to the conditions of greatest importance affecting the legibility of ordinary type: thus figures, capitals, italics, isolated letters, or letters in mathematical combinations are not considered. There is no very definite conclusion reached from the experiments, the order of merit differing appreciably from one test to another. The author suggests that the dominant factor is the familiarity of the reader with the type used, with the corollary that, unless the type is abnormal, the experiment ceases to be a test before sufficient material on which a judgment can be based has been accumulated.

Notwithstanding this conclusion there are several results recorded in the report which should prove of interest to a large number of readers. It is unfortunate that the report should have been allowed to appear in an inaccurate form—this is a case where the Stationery Office should issue a correcting slip without delay. The report contains several tables in which the different types are identified by number, and these numbers have been changed in the course of the report. For example, in Table 7 and Appendix 2 A, type No. 2 is "Modern" and No. 3 is "Old Face." In Table 10 the numbers or descriptions have been interchanged, and it is impossible to determine either from the tables or from the text where the mistake lies.

It is satisfactory to note from the preface that, though the present series of experiments has to a large extent failed to yield the information required, further work is to be carried out at some later time in connection with a general research into the physiology of vision. It is to be hoped that when such a report does appear it will be printed in type possessing those features which are considered to make for legibility. The only merit of the type selected for the present report is the very questionable one of being "condensed." The admirable example set in printing the report of the Committee appointed by the Treasury "to select the best faces of Type and Modes of Display for Government Printing" should be followed more frequently. As that Committee pointed out, it costs no more to set up type of good than of bad design.

ABSTRACTS OF PATENT SPECIFICATIONS

274593. Cox, H. N.

OPTICAL SYSTEMS.

In connection with an arrangement of prisms for producing a number of similar images, each prism is fitted with an adjusting device for correcting the deviation, consisting of a part of a plano-concave, and a part of a plano-convex lens, which are adjustable upon each other. When the correct adjustment is obtained, the projecting parts are ground off, and the whole mounted as a unit.

274616. Barr and Stroud, Ltd., and others.

BINOCULARS, &C.

The prism system is adjusted relatively to a holder before it is placed in position in the casing through a hole provided for the purpose. The holder can then be itself adjusted within the casing.

274777. Gerdes, A. F.

TELESCOPIC SIGHTS.

To convert a telescopic sight into a night sighting device, a sleeve carrying a lamp at the focus of a lens of short focal length is slipped on to the eyepiece. The lamp illuminates the graticule, and projects it upon the target.

274826*. Soc. Anon. des Charbons Actifs.

SPECTROSCOPES.

A spectroscope is employed for the analysis of gases, using the ultra-violet region, the lines being photographed or received upon a fluorescent screen. Quartz or fluorspar lenses constitute the optical system, and the tube is closed by quartz plates.

274842*. Grambach, C.

REFLECTORS.

In forming silvered reflectors, the rear edge of the plate is bevelled and roughened, and the protecting coating, but not the silvering, is carried across it.

275381. Barr and Stroud, Ltd., and another.

RANGEFINDERS.

An anti-friction screw gearing is employed to move the deviating prisms. The screw rotates in ball bearings, and engages a carriage provided with rollers arranged to fit the angle of the screw. The carriage, which slides upon guide rods, moves the prisms with it.

275389. Taylor, M. H., and others.

BINOCULARS.

An inner casing is moved by means of a yoke engaging the focussing screw to obtain the desired setting of the objectives. The eyepiece is carried by a cylindrical portion sliding upon the main section of the inner casing.

275400. Barr and Stroud, Ltd., and others.

RANGEFINDERS.

A rangefinder for determining altitudes is provided with an operating head controlled from a working head, and is fitted with a differential gear device so designed that a movement of the tube of the instrument does not affect the relative positions of the two heads.

275409. Sander, M.

SURVEYING.

A tachometer is provided with a series of rotary scales and sliders by which the distances and angles determined by the instrument are transferred to a pencil moving over a sheet of paper. In this manner a record of the survey is made.

* Specification published before acceptance.

R. J. T.

CORRESPONDENCE

LETTER RECEIVED BY DR R. S. CLAY

I see from *Nature* of 17th March, that you have delivered an address on the stereoscope before the Optical Society. I have not gone at all deeply into the history of the stereoscope but have had occasion lately to make some enquiries into the matter in writing a short biography of Mr J. B. Dancer who was formerly a manufacturing optician in this city.

From the evidence I have been able to collect there seems very little doubt that Dancer was the first one actually to produce a stereoscopic camera and apparently he was not only the first maker but the actual originator and inventor of the idea. This claim may not conflict with any of the statements made in your paper, but I thought you might like to know of the existence of this claim and also that the original camera, probably made about the year 1852, has recently been unearthed by his eldest son, Mr W. Dancer, who died here only yesterday at the age of 87 and with whom I was quite intimate. This camera has recently been presented to the Manchester Literary and Philosophical Society. I need not at the moment go over all the evidence as to priority except to say that the claim was undoubtedly recognised at the time by the Literary and Philosophical Society, of which Dancer was an active member, and I have also seen a great many letters from Sir David Brewster and others bearing on the point.

I may add, however, that Dancer never claimed to have been the originator of the idea of stereoscopic photography, but previously all these photographs had been taken by means of two cameras generally widely separated; and on the point of the correct separation of the lenses Dancer appears to have been almost alone in his advocacy of the inter-pupillary distance as the correct separation of the lenses in order to produce a true stereoscopic effect.

Dancer unfortunately failed to patent his camera through some technicality at the Patent Office and reaped little benefit from the invention except that he probably actually made and sold a large number of cameras. If you are interested further in the matter I shall be glad to give you any information, but I thought in justice to Dancer, whose later years were veiled in obscurity owing to blindness, some more public recognition of his invention should be given.

The above-mentioned camera by Dancer is the subject of a paper to be read at the International Congress of Photography in London in July.

HENRY GARNETT,
Messrs Flatters & Garnett, Ltd.
309 Oxford Road,
Manchester.

17th March, 1928.